CGTA 2023

Conference on Geometry: Theory and Applications

June 19 - 23, 2023

Schloss Weinberg, Kefermarkt, Austria

Scientific Committee:

Bert Jüttler, Hans-Peter Schröcker, Ivan Izmestiev

Institute of Applied Geometry JKU Linz

universität innsbruck Arbeitsbereich für Geometrie und Vermessung

TU WIEN TECHNISCHE UNIVERSITÄT WIEN Vienna | Austria





Abstracts CGTA 2023

1 Invited Talks

Legerský, Jan:

Flexibility of frameworks using edge colorings

Lubbes, Niels:

Singular loci and topology of surfaces containing two circles through each point

Manni, Carla:

Maximally smooth splines on triangulations

Springborn, Boris:

Discrete conformal maps

Žagar, Emil:

Optimal approximations by parametric polynomial curves and surfaces

Flexibility of frameworks using edge colorings

Jan Legerský[*,1]

[1] Czech Technical University in Prague
jan.legersky@fit.cvut.cz

A framework, which is a graph together with a realization of its vertices in the plane such that adjacent vertices are mapped to distinct points, is called flexible if it can be continuously non-trivially deformed while preserving the edge lengths, i.e., the distances between adjacent vertices; otherwise it is rigid.

It is well known that for a fixed graph being flexible/rigid is a generic property in the space of realizations. But even if a graph is generically rigid, it might admit non-generic flexible realizations. For instance, the complete bipartite graph on 3+3 vertices, which is generically rigid, has two families of flexible realizations given by Dixon.

A few years ago, we showed that a graph admits a (non-generic) flexible realization if and only if its edges can be colored surjectively by two colors so that each cycle is either monochromatic, or both colors occur at least twice. Such colorings are called NAC-colorings. In this talk, we focus on some contexts in which NAC-colorings have been considered: they characterize the existence of flexible realizations also for infinite graphs, their subclass is related to the flexibility on sphere and symmetric NAC-colorings determine the existence of flexes preserving rotational symmetry.

Moreover, for a class of frameworks consisting of triangles and parallelograms, the flexibility of a given framework is determined by the existence of a certain NAC-coloring. In this case, more detailed information about flexibility of a given framework can be obtained using the classes of an equivalence relation defined on the edge set of the graph. We illustrate the results on frameworks obtained from Penrose tilings and periodic tessellations by regular polygons.

Singular loci and topology of surfaces containing two circles through each point

Niels Lubbes

Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences
info@nielslubbes.com

It is classically known that a surface in $\mathbb{R}^3$ that contains two lines through each point is homeomorphic to either a cylinder or a plane. The Möbius analogues of doubly ruled surfaces are *celestial surfaces*, namely surfaces that contain two circles through each point. We address the following problem: *Classify celestial surfaces in the unit-sphere S^3 up to homeomorphisms.*

To solve this problem, we first classify the singular loci of celestial surfaces in terms of an algebro geometric invariant that encodes how a circle on a celestial surface meets the self-intersection locus as it moves in its pencil (see Figure 1). As an application, we find that almost all celestial surfaces are homeomorphic to one of 4 topological normal forms. We conjecture that any celestial surface is homeomorphic to one of 14 topological normal forms. This concerns ongoing work and we refer to [1] for partial results.

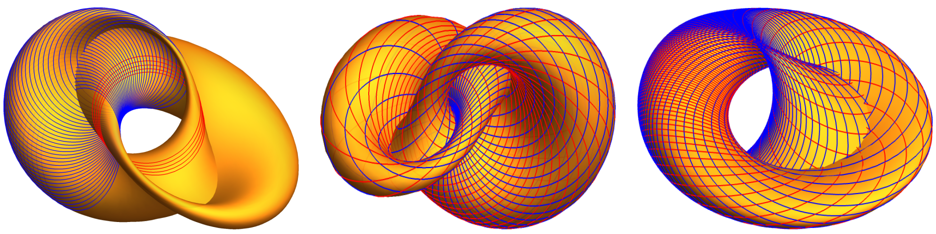


Figure 1: Stereographic projections of celestial surfaces with self-intersections.

In order to apply intersection theory, we embed celestial surfaces in a complex projective hyperquadric whose real points form S^3 (see [2]). Subgroups of the group of projective isomorphisms of this hyperquadric define non-Euclidean geometries and relates the singular loci of celestial surfaces to the factorization problem for bivariate quaternionic polynomials. In cooperation with Johanna Frischauf and Hans-Peter Schröcker, this geometric point of view led to a generalization of work by Rimas Krasauskas and Mikhail Skopenkov [3].

The classification of singular loci of celestial surfaces has also applications in computer vision. An ongoing cooperation with Josef Schicho concerns the camera calibration from a picture of a torus of revolution. Although such tori do not have real singularities, the singularities at complex infinity play an essential role.

References

[1] N. Lubbes, The shapes of surfaces that contain a great and a small circle through each point, arXiv:2205.14438, 2022.

[2] N. Lubbes, Translational and great Darboux cyclides. arXiv:1306.1917, 2022.

[3] M. Skopenkov and R. Krasauskas. Surfaces containing two circles through each point. Math. Ann., 373(3-4):1299-1327, 2019.

Maximally Smooth Splines on Triangulations

Carla Manni[*,1]

[1] Department of Mathematics, University of Rome Tor Vergata
Via della Ricerca Scientifica, 00133 Rome, Italy
manni@mat.uniroma2.it

Splines, in the classical sense of the term, are piecewise functions consisting of polynomial pieces glued together in a certain smooth way, usually by imposing equality of derivatives up to a given order. Besides their theoretical interest, splines find application in a wide range of contexts such as geometric modeling, signal processing, data analysis, visualization, and numerical simulation, just to mention a few. For many of these applications, a high smooth join between the different pieces is beneficial or even required.

In the univariate case, splines of maximal smoothness, i.e., piecewise polynomials of degree p with C^{p-1} joins, are probably the best known and most used splines. In fact, smoother splines give the same approximation order as less smooth splines of the same degree but involve fewer degrees of freedom and have less tendency to oscillate.

When moving to the bivariate setting and considering polygonal partitions, e.g., triangulations, maximal smoothness is still very appealing but becomes an arduous task to achieve. To obtain splines of high smoothness on general triangulations in a stable manner, sufficiently large degrees have to be considered. An alternative is to use lower-degree macro-elements that subdivide each triangle into a number of subtriangles (or more general subdomains).

Simplex splines are one of the most elegant generalizations of univariate B-splines to the multivariate setting. They enjoy a nice geometric interpretation and several properties such as smoothness and recursion, knot insertion and degree elevation formulas.

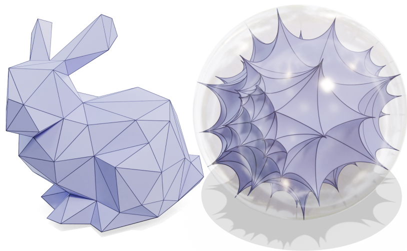
In this talk, after reviewing the main issues concerning the construction of highly smooth splines on triangulations, we consider a family of macro-elements of degree p and maximal smoothness $p-1$ and we discuss the construction of a suitable local representation for the related spline space in terms of simplex splines. In particular, we detail the important cases of C^2 cubic, and C^3 quartic macro-elements and we discuss several properties, such as local support, linear independence, and nonnegative partition of unity of the provided simplex spline basis.

The talk is based on a joint work with Tom Lyche and Hendrik Speleers.

Discrete Conformal Maps

Alexander Bobenko[1] Keenan Crane[2] Mark Gillespie[2]
Ulrich Pinkall[1] Peter Schröder[3] Boris Springborn[*,1]

[1] Technische Universität Berlin [2] Carnegie Mellon University [3] Caltech



[4]

This talk is about a discrete version of conformal maps. The story begins with an elementary definition of discrete conformal equivalence for triangle meshes. An unexpected connection with hyperbolic geometry leads to a rich mathematical theory [2] and efficient algorithms for applications in computer graphics and geometry processing [1]. In terms of applied mathematics, a unique feature of this theory is a canonical remeshing strategy for large deformations [4]. In terms of pure mathematics, one obtains discrete analogs of classical uniformization theorems [3].

References

[1] B. Springborn, P. Schröder, U. Pinkall: Conformal equivalence of triangle meshes, *ACM Trans. Graph.* 27:3 (2008), 77:1–77:11.

[2] A. Bobenko, U. Pinkall, B. Springborn: Discrete conformal maps and ideal hyperbolic polyhedra, *Geom. Topol.* 19:4 (2015), 2155–2215. arXiv:1005.2698 (2010).

[3] B. Springborn: Ideal hyperbolic polyhedra and discrete uniformization, *Discrete Comput. Geom.* 64:1 (2020), 63–108.

[4] M. Gillespie, B. Springborn, C. Crane: Discrete conformal equivalence of polyhedral surfaces, *ACM Trans. Graph.* 40:4 (2021), 103:1–103:20.

Optimal Approximations by Parametric Polynomial Curves and Surfaces

Emil Žagar*,[1]

[1] Faculty of mathematics and physics, University of Ljubljana, and Institute of mathematics, physics and mechanics,
Jadranska 19, 1000 Ljubljana, Slovenia
emil.zagar@fmf.uni-lj.si

Construction of parametric polynomial curves and surfaces is a fundamental task in computer–aided geometric design. It usually relies on given geometric data (points, tangent directions, curvatures,...) obtained as measurements or as samples from implicitly given (algebraic) objects. In the later case it is often important to reconstruct or approximate them by simple parametric polynomial counterparts.
In this talk we will consider optimal polynomial approximations of some fundamental curves and surfaces which do not possess exact polynomial representations (circular and elliptic arcs, spherical squares and rectangles, dots). The standard Hausdorff distance will be used as an error measure which makes the problem highly nonlinear. Some general approaches will be described and several numerical examples will be provided.

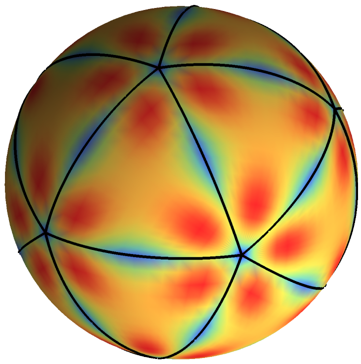


Figure 1: The optimal quartic G^2 spline approximant of the whole unit sphere together with the Gaussian curvature (red regions indicate higher curvature).

2 Minisymposia

2.1 Computer Aided Geometric Design. Organizer: Peter Salvi, Budapest University of Technology and Economics (Hungary)

Mann, Stephen:

A generalized blending scheme for arbitrary order of continuity between n-sided patches

Qin, Kaikai:

Blending Bézier patch for multi-sided surface modeling

Reif, Ulrich:

Modeling and simulation with watertight trimmed NURBS surfaces

Várady, Tamás:

Control point based multi-sided surfaces over curved, multi-connected domains

A Generalized Blending Scheme for Arbitrary Order of Continuity Between n-Sided Patches

Xiang Fang[1] Stephen Mann[*,2]

[1] University of Waterloo
x2fang@uwaterloo.ca

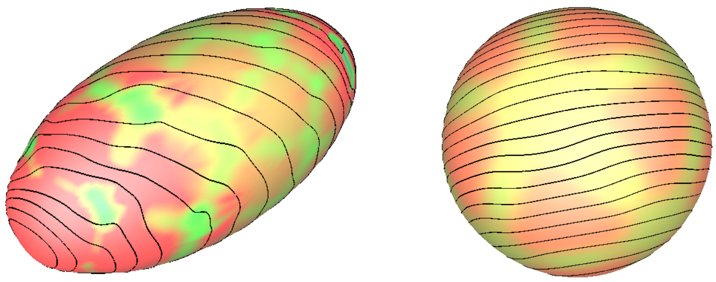
[2] University of Waterloo
smann@uwaterloo.ca

A blending scheme for a mesh with n-sided faces is derived for solving the scattered data interpolation problem. The algorithm takes a mesh as input, and the resulting surface interpolates the positions and derivatives of the mesh. The blending functions used in the algorithm contain parameters to allow the algorithm to construct surfaces with an arbitrary, user-specified order of continuity between neighboring patches.

The method proceeds by, for each mesh vertex V, constructing a *vertex surface* S_V that interpolates the vertex and its neighboring vertices, $V_0, V_1, \ldots, V_{n-1}$. The vertex surface S_V is split into n quadrangular *boundary patches* (containing V, V_i, and the remaining vertices of the two faces on either side of boundary between V and V_i). For an edge of the mesh (containing vertex V_A and V_B), there are two boundary patches corresponding to the vertex surfaces S_{V_A} and S_{V_B}; this pair of boundary patches is blended to obtain a *boundary surface* S_{AB}. For each n-sided face F, there is one boundary surface per edge of the face; these n boundary surfaces are split into n-sided patches, and these n patches are blended to obtain the final surface patch corresponding to F.

By using appropriate vertex surfaces, appropriate splitting of the vertex and boundary surfaces, and by using appropriate blending functions at each step, the resulting piecewise surface interpolates all the data at the vertices and the patches meet with continuity specified by the user.

Examples of G^2 surfaces constructed with this method are shown in Figure 1.



(a) (b)

Figure 1: G^2 surfaces shaded with Gaussian curvature, overlaid with isophotes; (a) Triangulated ellipsoid; (b) Dodecahedron.

References

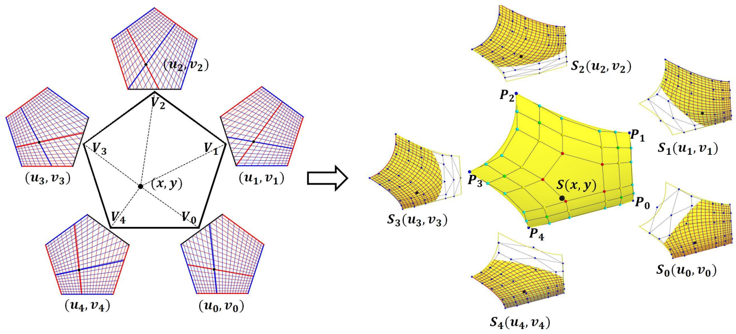
[1] X. Fang, A Generalized Blending Scheme for Arbitrary Order of Continuity, Ph.D. dissertation, University of Waterloo, 2023.

Blending Bézier patch for multi-sided surface modeling

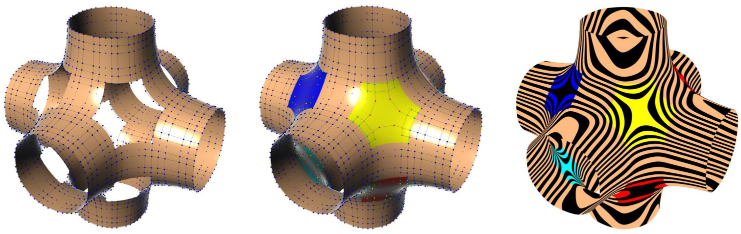
Kaikai Qin*,[1] Yajuan Li[1] Chongyang Deng[1]

[1] School of Science, Hangzhou Dianzi University
Hangzhou, Zhejiang Province, China
473397537@qq.com
liyajuan@hdu.edu.cn
dcy@hdu.edu.cn

This study proposes a new n-sided ($n \geq 4$) control point-based surface patch, the blending Bézier patch (BB patch) by constructing corner Bézier surface and using Gregory corner blending [1, 2]. A BB patch is defined on a regular polygonal domain with an n-sided control net generalized from quadrilateral Bézier patches, and it reduces to an ordinary Bézier patch when $n = 4$. There are two main steps for constructing a BB patch: defining a corner Bézier patch for each corner, and blending all corner Bézier patches using rational blending functions. Because the boundary behaviors of the BB patch are similar to those of the Bézier patch, a BB patch can be easily joined to surrounding Bézier or other BB patches. As an application, we used BB patches to fill the holes with G^2 continuity.



Corner blending Bézier patch



G^2 hole filling

References

[1] Gregory, J.A.; Hahn, J.M.: "A C2 polygonal surface patch." Computer Aided Geometric Design 6.1 (1989): 69-75.

[2] Salvi, P.; Várady, T.: "G2 surface interpolation over general topology curve networks." Computer Graphics Forum. 33.7 (2014): 151-160.

Modeling and Simulation with Watertight Trimmed NURBS Surfaces

Ulrich Reif[1]

[1] Fachbereich Mathematik, Technical University of Darmstadt
Schlossgartenstr. 7, 64289 Darmstadt, Germany
reif@mathematik.tu-darmstadt.de

Trimmed NURBS surfaces are the standard for industrial modeling of freeform geometry. However, current implementations typically yield gaps between neighboring surface patches along trimming curves. This issue causes problems for interactive design, surface quality, and simulation. We present a novel class of trimmed spline surfaces allowing for accurate boundary control to overcome these difficulties. It provides a framework to construct composite splines surfaces of arbitrary smoothness and is fully compatible with standard data exchange formats like IGES or STEP, thus solving an old problem in CAD.

Beyond the modeling of smooth surfaces, the approach opens new possibilities to parameterize domains bounded by splines, as requested for simulation. The notoriously difficult meshing problem, equally present in traditional and isogeometric FE techniques, is removed by the straightforward derivation of a single-patch parametrization of the domain from the boundary representation. We illustrate the potential of the method by showing first numerical results for elliptic PDEs with Dirichlet boundary conditions.

References

[1] F. Martin and U. Reif, Trimmed Spline Surfaces with Accurate Boundary Control. In: C. Manni and H. Speleers (eds.) Geometric Challenges in Isogeometric Analysis, Springer INdAM Series, 123-148 (2022).

Control point based multi-sided surfaces over curved, multi-connected domains

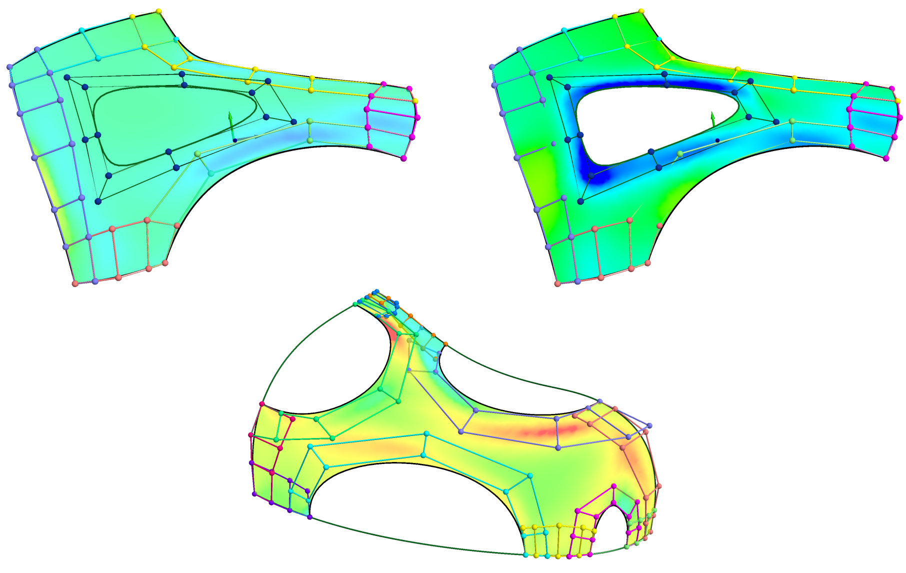
Tamás Várady

Budapest University of Technology and Economics
2. Magyar tudósok Blvd., 1117 Budapest, Hungary
varady@iit.bme.hu

We present a new control point based family of parametric surfaces to interpolate surface ribbons, i.e., boundary curves and cross-derivatives, given in Bézier or B-spline form. These patches are defined by a single surface equation and can represent complex, free-form geometries that smoothly connect to tensor-product Bézier or B-spline surfaces. The scheme is defined over a planar domain with curved edges that mimics the shape of the 3D boundary curves and are capable to handle strongly concave boundaries and periodic hole loops. These features can hardly be accomplished using representations based on polygonal domains.

We are going to present the basic ribbon concept and the composition of these surface patches. Important components of the construction include curved domain generation, methods to define local parameterizations, creating special blending functions associated with the control points of the ribbons, and options to edit the interior of the patch.

The most important applications include curve network based design, hole filling (vertex blending), general lofting and the approximate representation of trimmed surfaces, particularly when watertight connections are important. The strength of the scheme is its flexibility to define complex shapes in a natural manner; its main weakness is that it cannot be exported in standard form. Several examples will be given to compare the difficulties of classical surfacing approaches and the benefits of the new multi-sided scheme.



References

[1] T. Várady, P. Salvi, M. Vaitkus, Á. Sipos: Multi-sided Bézier surfaces over curved, multi-connected domains. Computer Aided Geometric Design 78 (2020), #101828.

[2] M. Vaitkus, T. Várady, P. Salvi, Á. Sipos: Multi-sided B-spline surfaces over curved, multi-connected domains. Computer Aided Geometric Design 89 (2021), #102019.

2.2 Differential Geometry.
Organizer: Gudrun Szewieczek, TU Berlin (Germany)

Brander, David:

New results on surfaces with harmonic Gauss maps

Lutz, Carl O. R.:

Canonical tessellations of decorated hyperbolic surfaces

Steinmeier, Jannik:

Discrete (constrained) elastic curves as invariants of Bäcklund transformations

Yasumoto, Masashi:

On timelike constant mean curvature surfaces in 3-dimensional Lorentz-Minkowski space

New results on surfaces with harmonic Gauss maps

David Brander *,[1]   Shimpei Kobayashi [2]

[1] Dept. of Appl. Math. and Comp. Sci., DTU - Technical University of Denmark
dbra@dtu.dk

[2] Dept. of Math., Hokkaido University
shimpei@math.sci.hokudai.ac.jp

A harmonic map from a Riemann surface into the 2-sphere can be used as the Gauss map for various types of special surfaces: e.g., surfaces of constant mean and Gauss curvature in Euclidean, hyperbolic, or spherical 3-space, as well as a spacelike maximal surface in the 3-dimensional Lorentzian Heisenberg group. In the last case, since the geometry is not isotropic, one must identify a specific direction in the 2-sphere (the vertical direction) with the center of the Lie algebra. The surface has singularities at points where the Gauss map is orthgonal to the vertical direction. In recent work [1], we found a simple relation between the geometry of the CMC surface in Euclidean space and the type of singularity of the maximal surface. Using this one can prove, for instance, that a maximal topological disc with singular boundary must have at least two cuspidal cross-caps on the boundary.

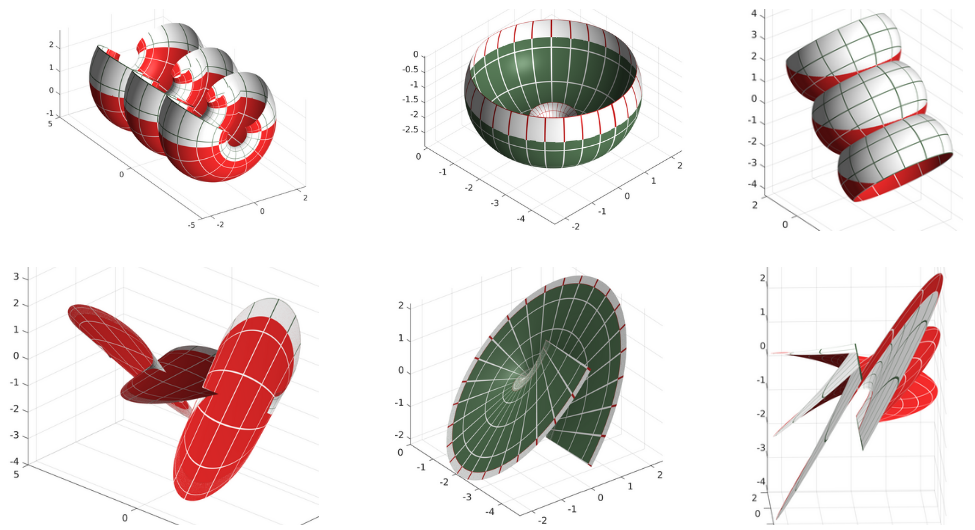


Figure 1: Top: Portions of a single constant mean curvature surface of revolution, with three different spatial orientations (three different choices of vertical direction). Bottom: the corresponding maximal surfaces in the Heisenberg group, in the same order.

References

[1] D. Brander and S. Kobayashi, *Maximal surfaces in the Lorentzian Heisenberg group*, arXiv:2302.10559v2[math.DG]

Canonical tessellations of decorated hyperbolic surfaces

Carl O. R. Lutz*[,1]

[1] Institute für Mathematik, Technische Universität Berlin
Straße des 17. Juni 136, 10623 Berlin, Germany
clutz@math.tu-berlin.de

Decorated hyperbolic cusp surfaces and their canonical tessellations are well-known objects with rich geometric structure [1]. They have proven to be an important tool for discrete uniformization theory and polyhedral realisation problems [3].

We will give an overview of a new generalisation of canonical tessellations to decorated finite type hyperbolic surfaces [2] (see Figure 1). Amongst others: a characterisation in terms of weighted distances and the hyperbolic geometric analogues of Delaunay's empty discs; connections to convex hulls in Minkowski space; and an analysis of the dependence of combinatorics on the decoration.

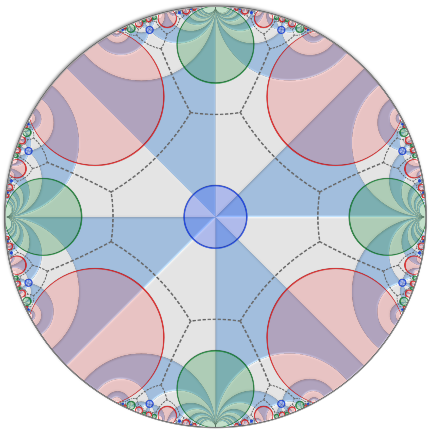


Figure 1: A tessellation of the hyperbolic plane corresponding to a weighted Delaunay triangulation (blue-grey chequered) and its dual weighted Voronoi decomposition (dashed lines) of a decorated hyperbolic surface. The surface can be obtained as the quotient of the hyperbolic plane by a Fuchsian group. It has a cone-point (blue), a cusp (green) and a flare (red).

References

[1] D. B. A. Epstein, R. C. Penner: Euclidean decompositions of noncompact hyperbolic manifolds, J. Differential Geom., 27(1), 1988, 67–80

[2] C.O.R. Lutz: Canonical tessellations of decorated hyperbolic surfaces, Geom. Dedicata, 217(14), 2023, 1–37

[3] B. Springborn: Ideal Hyperbolic Polyhedra and Discrete Uniformization, Discrete Comput. Geom., 64, 2020, 63–108

Discrete (constrained) elastic curves as invariants of Bäcklund transformations

Jannik Steinmeier*,[1]

[1] Lehrstuhl für Geometrie und Visualisierung - Technische Universität München
Boltzmannstraße 3, Garching bei München 85748, Germany
steinmei@ma.tum.de

Many flows of a discrete curve can be discretized by sequences of Bäcklund transformations. An example is the discrete smoke ring flow as in [2], [3]. We introduce an algorithm for the construction of discrete curves which stay rigid under such discrete flows. We take a closer look at the cases of two and three Bäcklund transformations which allow for the construction of elastic rods as in [1] and planar constrained elastic curves respectively. We discuss their geometric properties and compare them to their smooth counterparts.

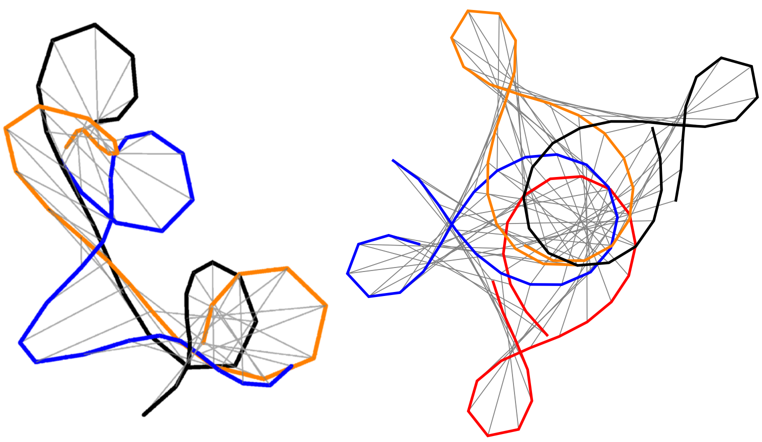


Figure 1: A discrete elastic rod in three dimensional space with two Bäcklund transformations (left) and a planar constrained elastic curve with three Bäcklund transformations (right)

This is based on joint work with Alexander Bobenko (TU Berlin, Germany), Tim Hoffmann (TU München, Germany), Andrew O. Sageman-Furnas (North Carolina State University, USA) and Gudrun Szewieczek (TU Berlin, Germany).

References

[1] A. Bobenko, Y. Suris: Discrete Time Lagrangian mechanics on Lie groups, with an application to the Lagrange top, Comm. Math. Phys, 204 (1999), 147-188.

[2] U. Pinkall, B. Springborn, S. Weissmann: A new doubly discrete analogue of smoke ring flow and the real time simulation of fluid flow, Journal of Physics A: Mathematical and Theoretical (2007), 40(42).

[3] T. Hoffmann: Discrete Hashimoto surfaces and a doubly discrete smoke-ring flow, In Discrete differential geometry (2008), Springer, 95-115.

On timelike constant mean curvature surfaces in 3-dimensional Lorentz-Minkowski space

Joseph Cho[1] Masashi Yasumoto[*,2]

[1] Institute of Discrete Mathematics and Geometry, TU Wien,
Wiedner Hauptstrasse 8-10/104, 1040 Wien, Austria
jcho@geometrie.tuwien.ac.at

[2]Graduate School of Technology, Industrial and Social Sciences,
Tokushima University, 2-1 Minamijyousanjima-cho, Tokushima 770-8506, Japan
yasumoto.masashi@tokushima-u.ac.jp

It is classically known that any conformal immersion into the Euclidean 3-space $\mathbb{R}^3$ with vanishing mean curvature can be described by a pair of holomorphic functions. Similarly, Kenmotsu [1] derived a representation formula for nonzero constant mean curvature (cmc for short) surfaces in $\mathbb{R}^3$ in terms of harmonic Gauss maps into the standard 2-sphere.

In this talk we discuss timelike cmc surfaces in Lorentz-Minkowski 3-space $\mathbb{R}^{2,1}$. Like in the case of minimal surfaces in $\mathbb{R}^3$, there is a Weierstrass-type representation for timelike minimal surfaces in terms of p-holomorphic functions. Also, there is a Kenmotsu-type representation for timelike nonzero cmc surfaces in $\mathbb{R}^{2,1}$ in terms of Lorentz harmonic Gauss maps into the de Sitter 2-sphere. Although a unified Kenmotsu-type representation was already derived by [2], we derive a Kenmotsu-type representation for timelike cmc surfaces in $\mathbb{R}^{2,1}$ based on integrable systems approach. Finally, if time allows, we introduce our attempt toward a integrable discretization of timelike cmc surfaces in $\mathbb{R}^{2,1}$. In Figure 1, examples of discrete timelike minimal surfaces in $\mathbb{R}^{2,1}$ based on our approach are shown.

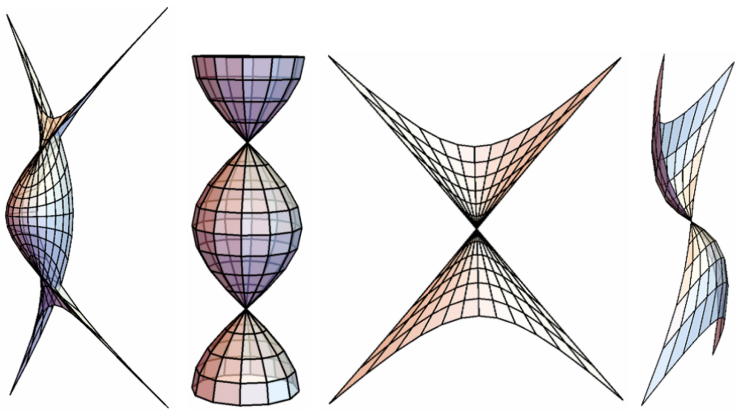


Figure 1: Examples of discrete timelike minimal surfaces in $\mathbb{R}^{2,1}$

References

[1] K. Kenmotsu, Weierstrass formula for surfaces of prescribed mean curvature, Math. Ann. 245 (1979), no. 2, 89-99.

[2] M. Kokubu, Application of a unified Kenmotsu-type formula for surfaces in Euclidean or Lorentzian three-space, preprint.

2.3 Algebraic Methods in Geometry. Organizer: Jan Vršek, University of West Bohemia, Plzeň (Czech Republic)

Alcázar, Juan Gerardo:

Affine equivalences of rational and analytic parametric curves in arbitrary dimension

Hoxhaj, Eriola; Menjanahary, Jean Michel:

Using algebraic geometry to reconstruct a Darboux cyclide from a calibrated camera picture

Krasauskas, Rimvydas:

Singularities of Dupin cyclidic cubes

Sendra–Arranz, Javier:

Recovery of plane curves from branch points

Sendra, J.Rafael:

Algebro-geometric methods to approach radical differential equations

Affine equivalences of rational and analytic parametric curves in arbitrary dimension

Juan Gerardo Alcázar*,[1]   Hüsnü Anıl Çoban[2]   Uğur Gözütok[2]

[1] Universidad de Alcalá (Madrid, Spain)
juange.alcazar@uah.es

[2]Karadeniz Technical Univeristy (Trabzon, Turkey)
hacobahn@ktu.edu.tr, ugurgozutok@ktu.edu.tr

We generalize the ideas in the paper [1], where projective equivalences of 3D curves are considered, to detect affine equivalences of parametric curves in arbitrary dimension. The algorithm is applicable not only to rational curves, but also to parametric curves, satisfying certain hypotheses, involving an analytic function (e.g. catenary curves, helixes, etc.) As in the case of [1], our algorithm leans on generating certain invariants, with closed form expressions. In turn, these invariants allow us to compute, via bivariate factoring, Möbius functions in the parameter space from which affine equivalences can be recovered.

References

[1] Gözütok U., Çoban, Sağıroğlu Y., Alcázar J.G. (2023), *A new method to detect projective equivalences and symmetries of rational 3D curves*, Journal of Computational and Applied Mathematics, Vol. 419, 114782.

Singularities of Dupin Cyclidic Cubes

Rimvydas Krasauskas[*,1] Eriola Hoxhaj[2] Jean Michel Menjanahary[1]

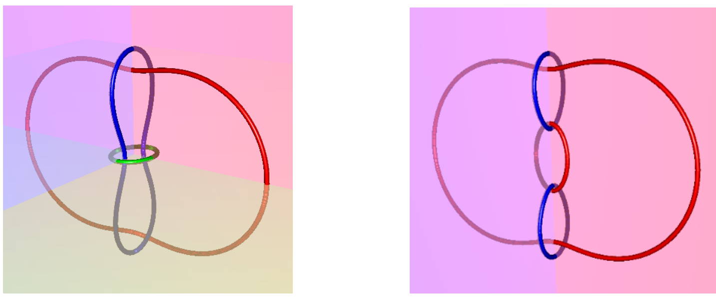
[1] Institute of Computer Science, Vilnius University
Didlaukio 47, 08303 Vilnius, Lithuania
rimvydas.krasauskas@mif.vu.lt

[2]Johannes Kepler University, Linz, Austria

A triply orthogonal coordinate system in $\mathbb{R}^3$ is called Dupin cyclidic (DC) if its coordinate lines are circles or straight lines. A Dupin cyclidic cube is a volume cut out of a DC coordinate system by certain couples of coordinate surfaces in all 3 directions. For various potential applications of DC cubes, it is important to avoid singularities. This was the main motivation for the current research.

Technically DC coordinate system is represented by the 3-linear rational quaternionic map $F : (\mathbb{R}P^1)^3 \to Im\mathbb{H} = \mathbb{R}^3$, $F = ND^{-1}$, $N,D \in \mathbb{H}[s,t,u]$ (see, e.g. [2]). The singular locus $sing(F) \subset \mathbb{R}^3$ of F is defined as the image of all points where its Jacobian vanishes. It is natural to consider DC coordinates F up to Möbius (i.e. conformal) equivalency in $\mathbb{R}^3$.

Theorem 1 *For any DC coordinates F, its singularities $sing(F)$ can be reduced to two main canonical forms that are arrangements of symmetric bicircular quartic curves placed on orthogonal planes:*
(Σ_1) three 1-oval curves (see Fig. left); (Σ_2) two 2-oval curves (see Fig. right).



Both forms Σ_1 and Σ_2 depend on two parameters and were considered by G. Darboux [1, p. 472] in the case of different coordinate systems. Note that in particular, the form Σ_2 can degenerate to the classical focal conics on two orthogonal planes.

Theorem 2 *The set of Möbius classes of DC coordinates is a disjoint union of three subsets:*
(A) the cases with a point where all 3 intersecting coordinate surfaces are planes (classified by their singularities of all possible forms Σ_1 and Σ_2);
(B) all other cases with at least 2 intersecting coordinate planes (all Σ_2 singularities);
(C) the case of offsets of a right circular cone (the singular locus is a line).

References

[1] G. Darboux, Principes de géométrie analytique, Gauthier-Villars, 1917.

[2] S. Zube, R. Krasauskas: Representation of Dupin cyclides using quaternions, Graphical Models 82 (2015), 110–122.

Using Algebraic Geometry to Reconstruct a Darboux Cyclide from a Calibrated Camera Picture

Eriola Hoxhaj*,[1] Jean Michel Menjanahary*,[2] Josef Schicho[3]

[1] RISC, JKU - Johannes Kepler University Linz
eriola.hoxhaj@risc.jku.at

[2] Institute of Informatics, Vilnius University Lithuania
jean.menjanahary@mif.vu.lt

[3] RISC, JKU - Johannes Kepler University Linz
josef.schicho@risc.jku.at

The problem of recognizing an algebraic surface from a single view apparent contour can be reduced to the recovering of a homogeneous equation in four variables from its discriminant. The problem is solved in 1992 by d'Almeida [1] for smooth surfaces. A generalization of d'Almeida algorithm on surfaces with at most ordinary singularities is considered in [2]. We introduce in [3] an algorithm reconstruction for remarkable surfaces called *Darboux cyclides* [4]. A Darboux cyclide doubly covers the absolute conic to either a nodal or a cuspidal singular curve. We use such remarkable properties to generate an algorithm reconstruction that applies to Darboux cyclides. The existing algorithms in [2] cannot be applied directly as a Darboux cyclide may have non-ordinary singularities, namely the cuspidal case and/or some isolated singularities.

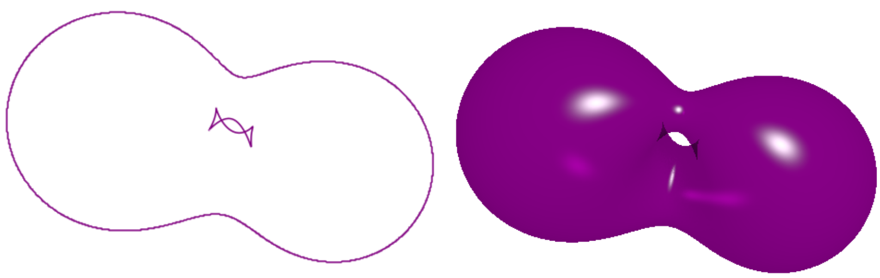


Figure 1: Apparent contour and a Darboux cyclide reconstructed.

References

[1] J. d'Almeida, Courbe de ramification de la projection sur $\mathbb{P}^2$ d'une surface de $\mathbb{P}^3$, Duke Mathematical Journal, 65 (1992), pp. 229–233.

[2] M. Gallet and N. Lubbes and J. Schicho and J. Vršek, Reconstruction of surfaces with ordinary singularities from their silhouettes, SIAGA, 3 (2019), pp. 472–506.

[3] E. Hoxhaj and J.M. Menjanahary and J. Schicho, Using Algebraic Geometry to Reconstruct a Darboux Cyclide from a Calibrated Camera Picture, https://arxiv.org/abs/2208.05209.

[4] H. Pottmann and L. Shi and M. Skopenkov, Darboux cyclides and webs from circles, Computer Aided Geometric Design, 29 (2012), pp. 77–97.

Recovery of Plane curves from Branch points

Javier Sendra–Arranz

Max Planck Institute for Mathematics in the Science, Leipzig
sendra@mis.mpg.de

The classical Hurwitz numbers counts the amount of simple branched coverings, of a fixed degree, from curves of fixed genus with a fixed branch locus. Analogously, plane Hurwitz numbers count the number of such branched coverings that are a linear projections from a plane curve. This number is only known for plane curves of degree 3 and 4, and it is 40 and 120 respectively. For these degrees, we study how to compute the 40 and 120 plane curves from the branch locus. Moreover, given a real branch locus we compute the number of real plane curves among these 40 and 120 curves. The results to be presented in the talk have been achieved jointly with D. Agostini, H. Markwig, C. Nollau, V. Schleis, and B. Sturmfels.

Algebro-geometric methods to approach radical differential equations

J.Rafael Sendra
Universidad de Alcalá,
Dpto. Física y Matemáticas,
Alcalá de Henares, Madrid, Spain

The basic idea of the so-called algebro-geometric approach for determining solutions of algebraic diferential equations is to assign an algebraic variety to the differential equation(s) to afterwards derive information on the solutions from the properties of the algebraic variety. If the coefficients of the equations are polynomials in the independent variables, the associated underlining variety is natural. But, how can we proceed is the coefficients are now nested radicals of polynomials? Illustrating examples of this are

$$4(y')^3\sqrt{x+1}^3-(x+1)yy'+y^2=0,$$

or

$$\left(-\sqrt{x_2}\,\frac{\partial y(\overline{x})}{\partial x_3}+2\,\frac{\partial y(\overline{x})}{\partial x_1}\right)\sqrt{x_1+\sqrt{x_2}}+2\sqrt{x_2}\,\frac{\partial y(\overline{x})}{\partial x_2}-y(\overline{x})^2-\frac{\partial y(\overline{x})}{\partial x_1}=0,$$

or even

$$\sqrt{x}\,y''-\sqrt{y^3}=0.$$

In this talk we will recall some basic facts on radical varieties, and we will present an algorithm to transform, if possible, a given ODE or PDE with radical function coefficients into one with rational coefficients by means of a rational change of variables so that solutions correspond one-to-one.

The main ideas presented in this talk have been jointly elaborated with my colleagues J. Caravantes, S. Falkensteiner, D. Sevilla, and C. Villarino and

1

2.4 Isogeometric Analysis.
Organizer: Deepesh Toshniwal, TU Delft (Netherlands)

Dijkstra, Kevin:

Local THB-spline Bézier projection

Hinz, Jochen:

PDE-Based parameterisation techniques for multipatch domains

Knez, Marjeta:

C^1 smooth isogeometric Clough–Tocher splines over a curved triangulation

Shakur, Emad:

IGA for solving multiphase engineering problems with precise and explicit interface representation

Local THB-Spline Bézier Projection

Kevin Dijkstra*,[1] Deepesh Toshniwal[2]

Delft Institute of Applied Mathematics, Delft University of Technology
Mekelweg 4, 2628 CD, Delft, Netherlands
k.w.dijkstra@tudelft.nl [2]d.toshniwal@tudelft.nl

In Isogeometric Analysis, the use of adaptive splines allows for significant savings when capturing solutions or approximating geometries with localized features. One such adaptive spline methodology that is popular for its simple implementation and attractive properties is that of Truncated Hierarchical B-splines (THB-splines) [1]. In this talk we will present a local THB-spline projector that is an extension of the Bézier projector introduced in [3]. The latter consists of two steps: an initial L^2 projection on every element, and a global reconstruction step where these local projections are combined to form a final global smooth spline. The first step utilizes local linear independence of B-splines on each element.

The property of local linear independence on every element is missing for THB-splines. Then, under assumptions on the mesh, we show that there exist collections of mesh elements such that the set of THB-splines with support on these collections are linearly independent over these collections. These collections are local in the sense that the elements in them are adjacent and their number depends only on the spline degree.

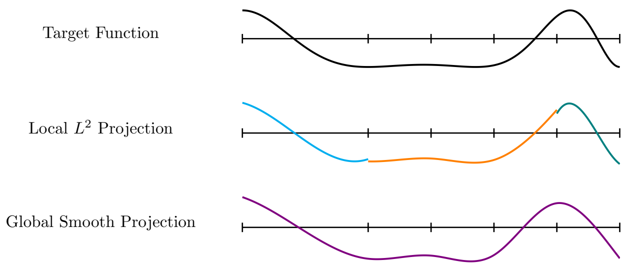


Figure 1: The THB-spline Bézier projector applied to the target. An initial local L^2 projection followed by a smoothing to construct a global spline.

The above allows us to formulate a Bézier projector for THB-splines. Optimal convergence is shown and we compare our Local THB-spline Bézier projector with [2]. Lastly, an adaptive refinement strategy is introduced to produce meshes that conform to our assumptions and a comparison with [2] is made.

References

[1] Carlotta Giannelli, Bert Jüttler, and Hendrik Speleers. Thb-splines: The truncated basis for hierarchical splines. *Computer Aided Geometric Design*, 29(7):485–498, 2012.

[2] Alessandro Giust, Bert Jüttler, and Angelos Mantzaflaris. Local (t)hb-spline projectors via restricted hierarchical spline fitting. *Computer Aided Geometric Design*, 80:101865, 6 2020.

[3] D. C. Thomas, M. A. Scott, J. A. Evans, K. Tew, and E. J. Evans. Bézier projection: A unified approach for local projection and quadrature-free refinement and coarsening of nurbs and t-splines with particular application to isogeometric design and analysis. *Computer Methods in Applied Mechanics and Engineering*, 284:55–105, 2015.

PDE-Based Parameterisation Techniques for Multipatch Domains

Jochen Hinz *,[1] Annalisa Buffa[2]

Chair of Numerical Modelling and Simulation,
École polytechnique fédérale de Lausanne (EPFL) Station Z CH-1015 Lausanne, Switzerland
[1]Jochen.Hinz@epfl.ch
[2]Annalisa.Buffa@epfl.ch

Isogeometric analysis (IGA) [1] is a variant of the finite element method (FEM) that employs NURBS for both the geometrical as well as the numerical aspects of the computational simulation workflow. Hereby, the surface-to-volume (StV) problem $\partial\Omega \to \Omega$ of finding a valid description of the geometry's interior from no more than a (spline / NURBS-based) description of its boundary becomes the IGA analogue of the classical meshing step. As such, the spline-based StV-step has received an increased amount of interest in recent years [2]. Besides providing a nondegenerate map (i.e., one with a strictly positive Jacobian determinant) proficient algorithms furthermore aim for numerically favourable parametric properties, such as orthogonal isolines and homogeneous cell sizes [3].
While substantial advancements have been made, most of them have been restricted to the singlepatch setting, i.e., when Ω can be parameterised from the unit quadrilateral. To expand upon this limitation, this talk proposes a framework for parameterising two- and *two-and-a-half*-dimensional domains that are topologically equivalent to some convex multipatch domain. For this, we adopt the concept of harmonic maps and propose several algorithms for tackling a PDE-based problem formulation that can be effortlessly integrated into a well-developed IGA software suite.
Parametric control is achieved by introducing an appropriate coordinate transformation in the parametric domain $\hat{\Omega}$. We discuss techniques for finding a so-called *control map* $\mathbf{s} : \hat{\Omega} \to \hat{\Omega}$ that induces a coordinate transformation capable of providing maps with boundary orthogonality, cell size homogeneity and various other desired features.
Finally, we discuss the generalisation to multiply connected domains.

References

[1] Hughes, Thomas JR, John A. Cottrell, and Yuri Bazilevs. "Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement." Computer methods in applied mechanics and engineering 194.39-41 (2005): 4135-4195.

[2] Xu, G., Mourrain, B., Duvigneau, R., & Galligo, A. (2011). Parameterization of computational domain in isogeometric analysis: methods and comparison. Computer Methods in Applied Mechanics and Engineering, 200(23-24), 2021-2031.

[3] Gravesen, J., Evgrafov, A., Nguyen, D. M., & Nørtoft, P. (2014). Planar parametrization in isogeometric analysis. In Mathematical Methods for Curves and Surfaces: 8th International Conference, MMCS 2012, Oslo, Norway, June 28-July 3 2012, Revised Selected Papers 8 (pp. 189-212). Springer Berlin Heidelberg.

C^1 smooth isogeometric Clough–Tocher splines over a curved triangulation

Marjeta Knez*,[1] Jan Grošelj[1]

[1] University of Ljubljana, Faculty of Mathematics and Physics
Jadranska 19, 1000 Ljubljana, Slovenia
marjetka.knez@fmf.uni-lj.si, jan.groselj@fmf.uni-lj.si

Bivariate polynomials of a fixed total degree naturally extend to splines over triangulations. The standard approach is to define these splines over a planar domain with a polygonal boundary that is partitioned by a collection of non-overlapping triangles. The spline function is then expressed locally over every triangle using barycentric coordinates and the Bernstein–Bézier representation. However, in the isogeometric setting, every triangle can be seen as a patch in the physical space, parameterized by a linear geometry mapping, defined by the first two barycentric coordinates, and the triangulation represents a multipatch domain. Replacing the linear geometry mapping by a polynomial mapping of a higher degree gives the curved triangles, where every edge of the curved triangle is parameterized by a polynomial curve. The curved triangles represent building blocks for so called curved triangulations that can be used to cover or approximate with more flexibility the domains with a smooth boundary.

Construction of smooth isogeometric spline spaces over a multipatch domain based on tensor product B-splines in the parametric domain has recently been a topic of an extensive research. Similar challenges occur also in the case of isogeometric splines over curved triangulations. In this work we focus on C^1 smooth isogeometric splines over a curved triangulation where every curved triangle is parameterized by a quadratic geometry mapping. The construction is based on a recently developed theoretical framework ([1]) for the analysis of C^1 smoothness conditions over the common interface of two quadratically parameterized mixed mesh elements. Since we want the construction to be local and independent of the geometry of the triangulation as well as to allows us to use, in the parametric space, the polynomials of not too high degree, we split every curved triangle into three curved micro-triangles by applying the Clough–Tocher refinement and observe the isogeometric spline space over the refined curved triangulation. Following the ideas used in [2], we define the interpolation problem that uniquely defines functions in this space and allows us to compute the basis. Knowing the basis opens the door for using these isogeometric spline spaces for different approximation problems as well as for solving different PDEs. Some of the numerical results confirming the theoretical findings and the properties of the derived spaces and their bases are presented at the end of the talk.

References

[1] J. Grošelj, M. Kapl, M. Knez, T. Takacs, V. Vitrih: C^1-smooth isogeometric spline functions of general degree over planar mixed meshes: The case of two quadratic mesh elements, submitted Feb. 2023.

[2] J. Grošelj, M. Knez: Generalized C^1 Clough–Tocher splines for CAGD and FEM, Computer methods in applied mechanics and engineering, May 2022, vol. 395, art. 114983.

IGA for solving multiphase engineering problems with precise and explicit interface representation

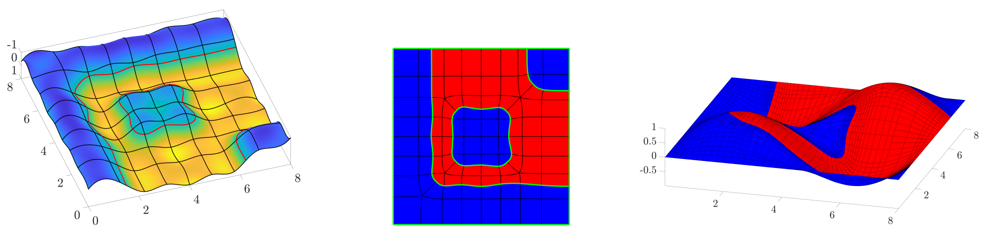
Emad Shakur

Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences, Altenberger Straße 69, 4040 Linz, Austria
E-mail: emad.shakur@oeaw.ac.at

Various engineering problems involve multiple material phases to describe their physical behavior. Examples include elasticity problems with multiple material properties, contact problems, and two-phase flow or fluid-structure interaction problems. Multiphase problems are usually sensitive to the quality of modeling the interface between the different material phases, where an inaccurate representation of the interface may lead to undesired inaccuracies in the solution of the physical response. Correspondingly, one of the current challenges in solving multiphase problems is to adopt more accurate numerical analysis approaches that allow for an accurate resolution of the interface between material phases.

The vast majority of studies in this field rely on finite element analysis (FEA). This requires representing the original structure geometry with an approximate representation that may lead to inaccurate estimations of physical measures, especially along the interface between the phases. So far, studies on this topic rarely adopt isogeometric analysis (IGA). The key point of IGA is that the analysis is performed over the original structure geometry where the structural boundaries are accurately represented. In principle, this could make IGA suitable for multiphase engineering problems with accurate interface representation between the material phases.

The main goal of the proposed approach is to allow for accurate interface representation within multiphase engineering problems. In the current work, the interfaces of the multiphase domain are determined based on the level-set approach. The level-set-based topology is mapped into a spline-based representation that is defined on an unstructured mesh following untrimming techniques. The mapped topology is characterized by an explicit and accurate interface representation which is exactly used for the analysis stage (IGA). For the purpose of incorporating this approach in moving interface problems, a consistent sensitivity analysis is developed. In addition, the proposed approach allows for controlling the continuity of the solution field along the interface while precisely preserving the discontinuous multiphase domain. The suggested approach has been successfully tested on various multiphase problems, including static linear elasticity, two-phase flow, and fluid-structure interaction.



(a) (b) (c)

Figure 1: (a) A level-set function. (b) The corresponding geometrical model after untrimming. Precise and explicit interfaces are represented using cubic B-spline curves. (c) The solution field of Poisson's equation over the multiphase domain.

3 Contributed Talks

Preparing Non-conforming meshes of B-Rep NURBS for DG-BEM simulations

Abdullah Ali*,[1] Benjamin Marussig[1]

[1] Institute of Applied Mechanics, Graz University of Technology
Technikerstraße 4, 8010 Graz, Austria
abdullah.ali@tugraz.at

Accommodating the flexibility of B-Rep NURBS with the strict meshing requirements of downstream applications such as numerical analysis involves numerous issues. Generating a valid volume mesh and trimming on NURBS, in particular, present severe challenges [1, 2].

The discontinuous Galerkin boundary element method (DG-BEM) allows for simulations on non-conforming surface meshes [3], alleviating the need for volumetric discretization. Furthermore, the non-conforming meshes reduce requirements on trimmed NURBS. Yet, non-conforming boundaries of the mesh must provide some connectivity information for the DG-BEM simulation. The algorithm developed in this work automatically determines where to join the NURBS patches and subsequently generates the connectivity information on associated meshes.

Several test examples, including models from our industrial partner GIPRO, demonstrate the performance of our approach. In addition, subsequent electrostatic simulation results are shown.

References

[1] Marussig B., Hughes T.J.R. "A Review of Trimming in Isogeometric Analysis: Challenges, Data Exchange and Simulation Aspects." Archives of computational methods in engineering, vol. 25 (4), 1059–1127, 2018

[2] Zhang Y.J. Geometric Modeling and Mesh Generation from Scanned Images. Chapman and Hall/CRC, New York, 2016

[3] Heuer N., Meddahi S. "Discontinuous Galerkin hp-BEM with quasi-uniform meshes." Numerische Mathematik, vol. 125 (4), 679–703, 2013

Analytic Signed Distance Representation of Polygons

Csaba Bálint*,[1] Gábor Valasek[1] Róbert Bán[1]

[1] Faculty of Informatics, Eötvös Loránd University

Distance representations are often employed in high-quality text rendering, collision detection, 3D printing, additive manufacturing, and real-time graphics effects. Analytic signed distance function (SDF) representations are often limited to representing a simple constructive solid geometry scene made from basic primitives, as complex scenes become expensive to evaluate, and SDFs are not closed under the min/max implementation of set-theoretic operations, that is, the results are only SDF lower bounds with deteriorating accuracy. On the other hand, discrete representations scale better in evaluation time and accuracy but consume a significant amount of memory and require possibly expensive generation steps.

In contrast, our method aims for relatively low memory consumption and the exact representation of analytic SDFs; however, it is only applicable to sets of polygons in two dimensions. Recently, we [1] proposed to augment the Voronoi diagram of polygons with elementary distance functions to represent the exact SDF of the input. However, we only approximated the vertex and edge Voronoi regions with convex bounding polygons such that the overlaps between the regions are slim. The SDF is trivial for vertex and edge regions and can be efficiently rendered, yet the exact Voronoi regions of the input polygons were not constructed explicitly in [1].

In this research, we note that quadratic Bézier splines can represent the region boundaries, and we present a purely geometric construction algorithm. Even though our proposed method requires parabola-parabola intersections, we show that these can be resolved by quadratic root finding instead of quartics due to the particular geometric configurations arising in this problem. Figure 1 shows an example configuration. We demonstrate that this also improves the robustness of the Voronoi diagram and median axis computations.

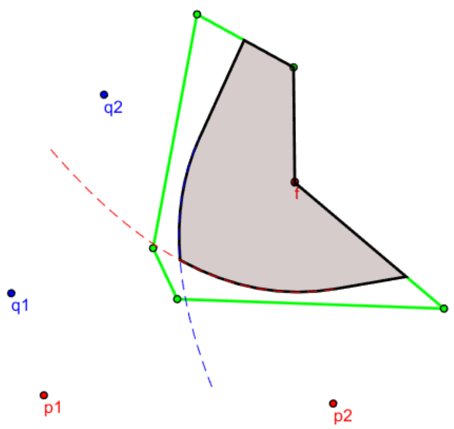


Figure 1: A vertex region starting out as a polygon (green) is cut with two parabolas (red and blue) with a common focus $\boldsymbol{f}$ defined by directrices between $\boldsymbol{p}_1, \boldsymbol{p}_2$ and $\boldsymbol{q}_1, \boldsymbol{q}_2$ points.

References

[1] Csaba Bálint, Gábor Valasek, and Róbert Bán. "Exact Signed Distance Function Representation of Polygons." Computer-Aided Design and Applications 20, no. 5 (2023): 1029–1042. doi: 10.14733/cadaps.2023.1029-1042.

Lipschitz Tracing on Discrete Fields

Róbert Bán*,[1] Gábor Valasek[1]

[1] Faculty of Informatics, Eötvös Loránd University

The main challenge in the visualization of general implicit surfaces is to calculate the ray-surface intersections robustly and efficiently on the GPU. Robust algorithms exist for special implicit functions. For example, sphere tracing [1] works on signed distance functions and their lower estimates. A recent work [2] uses local Lipschitz bounds to guarantee safe steps along the ray. The main drawback of their work is that Lipschitz segment tracing is only applicable to a subset of continuous implicit functions.

In practical applications, however, these functions are often stored in a discrete representation, usually as a regular grid of sample values. We present a generalization of Lipschitz segment tracing to discrete data filtered by tensor product interpolation. This relies on the fact that we can upper bound the maximum of the derivative of tensor-linearly filtered functions, as they are Bézier constructs. This can be further extended to rendering heightfields defined by higher degree Bézier patches.

References

[1] J.C. Hart, Sphere Tracing: A Geometric Method for the Antialiased Ray Tracing of Implicit Surfaces, *The Visual Computer 12 (1996), 527–545.* doi:10.1007/s003710050084

[2] E. Galin, E. Guérin, A. Paris, A. Peytavie: Segment Tracing Using Local Lipschitz Bounds, *Computer Graphics Forum 39 (2020), 545–554.* doi:10.1111/cgf.13951

On splitting spherical triangles into quadratic subpatches

Michal Bizzarri[*,1] Miroslav Lávička[1] Jan Vršek[1] Michael Bartoň[2]
Jiří Kosinka[3]

[1] Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 301 00 Plzeň, Czech Republic
bizzarri@kma.kma.cz, lavicka@kma.zcu.cz, vrsekjan@kma.zcu.cz

[2] Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Basque Country, Spain
mbarton@bcamath.org

[3] Bernoulli Institute, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands
j.kosinka@rug.nl

Various interpolation and approximation methods arising in several practical applications in geometric modelling deal, at a particular step, with the problem of computing suitable rational patches (of low degree) on the unit sphere. Therefore, we are concerned with the construction of a system of spherical triangular patches with prescribed vertices that globally meet along common boundaries. We investigate several configuration scenarios and propose methods leading to spherical macro-elements of the lowest possible degree. A potential applicability of our method is for instance any algorithm in which the construction of (e.g. interpolation) surfaces starts from rational normal vector fields.

Vector Products and Combinations of Points, Lines, and Planes

Bertold Bongardt*,[1] Friedemann Groh[2]

[1] Technische Universität Braunschweig, Institut für Robotik und Prozessinformatik, Mühlenpfordtstraße 23, 38106 Braunschweig, Germany, `b.bongardt@tu-bs.de`

[2] ISG Industrielle Steuerungstechnik GmbH, Gropiusplatz 10, 70563 Stuttgart, Germany, `friedemann.groh@isg-stuttgart.de`

In the case that a robot is constructed as a system of rigid bodies, its kinematics can be described by a *set of copies* of the Euclidean space, briefly called a *Euclidean system*, in which each Euclidean space is attached to one of the rigid bodies [3, 5]. Since "a rigid body can be considered as an assemblage of points, or planes, or lines, or described as a combination of all three of them" [2], the relative constraints and displacements between the different bodies can be described with respect to points, planes, lines, and their combinations. Next to linear transforms for points, planes, and lines, several parameterizations for combinations of them – especially for a *point on a line*, a *pointed line* – have been developed [4, 9].

Within the context of efficient kinematics computations, it has been observed [7] that next to the direction $\mathbf{n}$ and the moment $\mathbf{a} \times \mathbf{n}$ of a line, also the scalar product of anchor and direction $\mathbf{a}^\top\mathbf{n}$ as well as the expression $2 \cdot \mathbf{a} \cdot (\mathbf{a}^\top\mathbf{n}) - (\mathbf{a}^\top\mathbf{a}) \cdot \mathbf{n}$ only depend in linear manner on a rotation about an axis in space. Due to this characteristic, these terms have been classified as *Lee–Liang products* [7].

Geometrically, the action of the term $2 \cdot \mathbf{a} \cdot (\mathbf{a}^\top\mathbf{n}) - (\mathbf{a}^\top\mathbf{a}) \cdot \mathbf{n}$ corresponds to a rotation of π radians of the vector $\mathbf{n}$ about the vector $\mathbf{a}$, which is also called a *half-turn* [6]. For a unit $\mathbf{a}$ with $\|\mathbf{a}\| = 1$, the term equals the *negative Householder reflection* and the *sandwich product* $\mathbf{a}\,\mathbf{n}\,\mathbf{a}^{-1}$ with the geometric product [8]. In the planned conference presentation, geometric and algebraic properties of the π-rotation shall be analyzed, in particular with regard to the potential for modeling constraints and displacements in rigid body systems in a coherent and favorable manner. For this purpose, the *elemental triangle* of points, lines, and planes in Euclidean space shall be used as a navigation map [1, 10].

References

[1] L. Locher-Ernst. *Urphänomene der Geometrie*. Verlag am Goetheanum, 1937.

[2] K. H. Hunt and I. A. Parkin. "Finite displacements of points, planes, and lines via screw theory". In: *Mechanism and Machine Theory* 30.2 (1995).

[3] S. Stramigioli, B. Maschke, and C. Bidard. "A Hamiltonian formulation of the dynamics of spatial mechanisms using Lie Groups and Screw Theory". In: *Symposium Commemorating the Legacy, Works, and Life of Sir Robert Stawell Ball* (2000).

[4] J. M. Selig. *Geometric Fundamentals of Robotics*. Springer, 2005.

[5] M. Ceccarelli and T. Koetsier. "Burmester and Allievi: A Theory and Its Application for Mechanism Design at the End of 19th Century". In: *Journal of Mechanical Design* 130 (2008).

[6] J. M. Selig and M. Husty. "Half-turns and line symmetric motions". In: *Mechanism and Machine Theory* 46.2 (2011).

[7] F. Groh, K. Groh, and A. Verl. "On the inverse kinematics of an a priori unknown general 6R-Robot". In: *Robotica* 31.3 (2013).

[8] M. E. Horn. "Sandwich Products and Reflections". In: *Didaktik der Physik – Frühjahrstagung* (2015).

[9] G. Nawratil. "Point-models for the set of oriented line-elements – a survey". In: *Mechanism and Machine Theory* 111 (2017).

[10] C. Gunn. "A bit better: Variants of duality in geometric algebras with degenerate metrics". In: *arxiv.org* (2022).

Wall-Crossings Between Classes of Real Affine Cubic Surfaces

Jose Capco*,[1] Christoforos Spartalis[2]

Unit Geometry and Surveying, University of Innsbruck
Technikerstr. 13, 6020 Innsbruck, Austria
[1]jose.capco@uibk.ac.at

[2]christoforos.spartalis@uibk.ac.at

We present the classification of smooth real affine cubic surfaces in $\mathbb{P}^3(\mathbb{R})$ that are transversal at infinity based on a main result of a recent work by Finashin and Kharlamov [1]. Wall-crossings between such affine real cubic surfaces are discussed. We include the case in which the curves at infinity between two classes have different number of connected components. This allows us to investigate the connectivity graph of a generic pencil of cubics. We then discuss how these results may be used to investigate the number of connected components of the singularity-free space of a generic 6-SPS platform.

References

[1] **S. Finashin, V. Kharlamov**, *On Affine Real Cubic Surfaces*. Online: `https://arxiv.org/abs/2208.09502`, Accessed: 02.2023

Fast evaluation of derivatives of polynomial and rational Bézier curves

Filip Chudy*,[1]   Paweł Woźny[2]

[1] Dept. of Mathematics, Alma Mater Studiorum Università di Bologna, Italy
Institute of Computer Science, University of Wrocław, Poland
filip.chudy@unibo.it

[2] Institute of Computer Science, University of Wrocław, Poland
pawel.wozny@cs.uni.wroc.pl

A linear-time algorithm for computing a point on a polynomial or rational curve in Bézier form with good geometric and numerical properties has been recently given in [3]. This approach has also found applications in accelerating the evaluation of Bézier surfaces and even B-spline curves. In this talk, this result is applied in order to accelerate the evaluation of the first r derivatives of Bézier curves.

For the evaluation of the first r derivatives of a polynomial Bézier curve P_n of degree n with control points in a d-dimensional space, i.e., $P_n'(t), P_n''(t), \ldots, P_n^{(r)}(t)$ ($t \in [0,1]$) a method with $O(rnd)$ complexity is given.

For a rational Bézier curve R_n, an acceleration of the formulas for the first two derivatives given in [2] will be shown, as well as a geometric algorithm for higher derivatives, finding $R_n'(t), R_n''(t), \ldots, R_n^{(r)}(t)$ ($t \in [0,1]$) with $O(rnd)$ complexity.

References

[1] F. Chudy, P. Woźny: Fast evaluation of derivatives of polynomial and rational Bézier curves, 2023, *in preparation*.

[2] M.S. Floater: Derivatives of rational Bézier curves, Computer Aided Geometric Design 9 (1992), 161-174.

[3] P. Woźny, F. Chudy: Linear-time geometric algorithm for evaluating Bézier curves, Computer Aided-Design 118 (2020), 102760.

Topology of Quartic Loci Resulted From Lines Passing through a Fixed Point and a Conic

Thierry (Noah) Dana-Picard[*,1] Zoltán Kovács[2] Wei-Chi Yang[3]

[1] Jerusalem College of Technology, Israel
ndp@jct.ac.il

[2] The Private University College of Education of the Diocese of Linz, Austria
zoltan@geogebra.org

[3] Radford University, VA, USA
wyang@radford.edu

We investigate the topology of curves obtained as geometric loci via the following problem, inspired by college entrance exams in China.

Let be given a fixed point $A = (a_1, a_2)$, a curve $\mathscr{C}$ and lines passing through this fixed point and intersecting $\mathscr{C}$ at two points C and D. The locus point (the tracer) E, we are interested in finding, is lying on CD and satisfies $\overrightarrow{ED} = s \cdot \overrightarrow{CD}$ where s is a given real number.

In [1], parametric equations are computed. In this work, $\mathscr{C}$ is a conic, whose equation is $x^2 + qy^2 = 1$, $q \in \mathbb{R}$. We compute the implicit equation of the locus of E, showing that it is a quartic, that we denote by $c(a_1, a_2, q, s; x, y)$ and we study its topology. The main tools are elimination via Gröbner bases [2] and the derivatives of the obtained implicit equation.

By that way, we can set up a system of equations for the case of non-smooth curves. The loci are presented as a 1-parameter family of plane curves. By applying comprehensive Gröbner systems, we manage to obtain that the topology is changing between the drop-form (a piriform quartic in some special cases) and the analemma-form at

$$s = \frac{a_1^2 + qa_2^2 + 2\sqrt{a_1^2 + qa_2^2} + 1}{a_1^2 + qa_2^2 - 1},$$

by using the computer algebra system *Singular* and its `grobcov` package (which provides an effective implementation of the Kapur-Sun-Wang algorithm [3]).

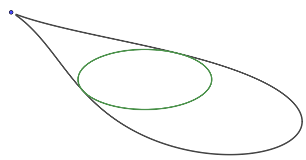


Figure 1: The fixed point $A = (-2, 1)$ and the ellipse for $q = 5$ imply the border case $s = 2$

References

[1] W.C. Yang: Locus Resulted from Lines Passing through a Fixed Point and a Closed Curve, The Electronic Journal of Mathematics and Technology 14(1) (2020), 1–17.

[2] D.A. Cox, J. Little, D. O'Shea: Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra. Springer (1992)

[3] D. Kapur, Y. Sun, D.K. Wang: A New Algorithm for Computing Comprehensive Groebner Systems, Proceedings of ISSAC'2010, ACM Press (2010), 29–36.

Discrete Geometry: Applications of Stair Convexity

Noa de-Roos[*,1]   Dr. Gabriel Nivasch[2]

[1]Ariel University, Israel

[2]Ariel University, Israel
https://www.gabrielnivasch.org

We consider two problems in discrete geometry, the **weak epsilon-nets** problem, and the **stabbing simplices** problem. Both problems have been extensively investigated. We improve their bounds and investigate cases that have not been explored yet.

In the first problem, which we study in the planar case, our objective is to construct a set of points $S \subset \mathbb{R}^2$ such that for any k points in $\mathbb{R}^2$, there exists a convex set that contains at least $\varepsilon|S|$ points of S, and does not contain any of the k points. We want $\varepsilon = \varepsilon(k)$ as large as possible.

In the second problem, the construction we want is a set of n-points $S \subset \mathbb{R}^d$ such that for any k points in $\mathbb{R}^d$ there are at most $c_{d,k}n^{d+1} + o(n^{d+1})$ simplices spanned by S that contain at least one of the k points ($0 < c_{d,k} < 1$). We search for a construction that gives us $c_{d,k}$ as small as possible. Also here, we focus on the case $d = 2$. This problem has not been studied before for k>1, as far as we know.

For these constructions we use a tool called *stair-convexity* (first introduced by Bukh, Matoušek and Nivasch, 2011 [1]) , which is a modification of the notion of convexity.

We present a construction which proves $\varepsilon(2) \geq 4/7$ (matching the known lower bound), and construction which provides strong evidence for $\varepsilon(3) \geq 0.47778\ldots$ (improving the previous bound of $\varepsilon(3) \geq 0.4545$). For the second problem, we present a construction which provides strong evidence for $c_{2,2} \leq 0.05297$ and $c_{2,3} \leq 0.06557$.

Finally, we consider the problem of constructing for every fixed k and every n, an n-point set $S \subseteq \mathbb{R}^2$ in general position and a k-point set $K \subseteq \mathbb{R}^2$ such that K stabs $\beta'_{d,k}n^{d+1} + o(n^{d+1})$ **stair-simplices** spanned by S for $\beta'_{d,k}$ as large as possible. The corresponding problem for regular simplices instead of stair-simplices has not been studied before for $k > 1$, as far as we know.

We present a proof of $\beta'_{2,2} = \frac{1}{16}$, and $\beta'_{2,k} \geq \frac{k(k+1)}{6(k+2)^2}$ and we conjecture this bound is tight.

References

[1] Boris Bukh and Jiří Matoušek and Gabriel Nivasch, 2011. Lower bounds for weak epsilon-nets and stair-convexity, Israel Journal of Mathematics 182, 199-228.

Line simplification with sequence of best approximations

Gábor Fábián*,1

[1] Department of Numerical Analysis, ELTE - Eötvös Loránd University Budapest
Pázmány Péter sétány 1/C, 1117 Budapest, Hungary
insomnia@inf.elte.hu

Line simplification and polygonal approximation are intensively studied topics, thanks to the usability of these kind of algorithms in e.g. cartography, digital image processing and pattern recognition. In this problem a polyline, i.e. a piecewise linear curve is given, defined by a lot of vertices, and we try to find an approximating polyline, that expresses more or less the same geometry, but with fewer vertices. Maybe the most popular line simplification methods are the Ramer-Douglas-Peucker and the Visvalingam-Whyatt algorithms [1] [2] [3]. These and many other similar methods use the vertices of the initial polyline as a starting point, thereafter iteratively omit the vertices causing only a negligible change in the geometry with respect to a given tolerance. Although, the simplified polyline should only be a good approximation of the initial one, these algorithms are not just approximation but interpolation methods, which in certain cases seems to be too strong restriction.

In this paper we show a novel approach to solve these kind of approximation problems. In our method the polyline is considered to be a parameterized plane curve, which has a piecewise linear parameterization. We will show, this parameterized curve is an element of a finite dimensional Hilbert-space equipped with a certain inner product. Our algorithm generates an increasing sequence of closed subspaces of the initial space, in which the best approximations can be easily computed. Therefore, our algorithm can be parameterized by the number of vertices of the approximating polyline, which is a virtue in cases when the computational cost is essential. For a given number of vertices the best approximation is calculated that minimizes some kind of squared distance between the approximating and the initial curve. We analyze the time complexity of the proposed algorithm and compare our results against the classical methods. Finally, we show a possible application of our algorithm in 2D collision detection.

References

[1] U. Ramer: An iterative procedure for the polygonal approximation of plane curves, Computer Graphics and Image Processing, Volume 1, Issue 3 (1972), 244-256.

[2] D. H. Douglas, T. K. Peucker: Algorithms for the reduction of the number of points required to represent a digitized line or its caricature, Cartographica: The International Journal for Geographic Information and Geovisualization, Volume 10, Number 2 (1973) 112-122.

[3] M. Visvalingam, J. D. Whyatt, Line generalisation by repeated elimination of points, The Cartographic Journal, Volume 30, Number 1 (1993), 46-51.

A symbolic analysis of the intersection curve between a torus and an ellipsoid

Jorge Caravantes[1] Gema M. Diaz-Toca[2] Mario Fioravanti[3]

Laureano Gonzalez-Vega[*,4]

[1]Universidad de Alcala, Spain
jorge.caravantes@uah.es

[2]Universidad de Murcia, Spain
gemadiaz@um.es

[3]Universidad de Cantabria, Spain
mario.fioravanti@unican.es@unican.es

[4]CUNEF Universidad, Spain
laureano.gonzalez@cunef.edu

Computing the intersection curve of two surfaces is a central problem in many areas, such as the CAD/CAM treatment of complicated shapes, the design of three-dimensional objects, computer animation, etc. For surfaces with a prescribed structure, extensive literature has described algorithms for solving the surface intersection problem in a very efficient manner by using the particular structure of the considered surfaces, such as ruled, revolution, canal or torus (see for example [2]). Quadrics are the simplest curved surfaces used in many applications, and computing their intersection is a practically relevant problem to solve.

We apply the approach introduced in [1] to determine the intersection of two quadrics to study the intersection curve between a torus $\mathscr{T}$ presented by the equation

$$f(x,y,z) = z^4 + \left(2R^2 - 2r^2 + 2x^2 + 2y^2\right) z^2 + R^4 - 2R^2r^2 - 2R^2x^2 - 2R^2y^2 + \\ + r^4 - 2r^2x^2 - 2r^2y^2 + x^4 + 2x^2y^2 + y^4 = 0$$

and an ellipsoid $\mathscr{E}$ presented by the equation $g(x,y,z) = z^2 + q_1(x,y)z + q_0(x,y) = 0$. The general case can be reduced easily to this case by a lineal change of coordinates.

We project the the intersection curve of $\mathscr{T}$ and $\mathscr{E}$ on the region corresponding to the projection of $\mathscr{T}$ and $\mathscr{E}$

$$\mathscr{D} = \left\{(x,y) \in \mathbb{R}^2 : \Delta_{\mathscr{E}}(x,y) \geq 0, \Delta_{\mathscr{T}_1}(x,y) \leq 0, \Delta_{\mathscr{T}_2}(x,y) \leq 0\right\}$$

where $\Delta_{\mathscr{E}}(x,y) = q_1(x,y)^2 - 4q_0(x,y)$, $\Delta_{\mathscr{T}_1}(x,y) = x^2 + y^2 - R^2$ and $\Delta_{\mathscr{T}_2}(x,y) = x^2 + y^2 - R^2$. This intersection curve is defined implicitely by the resultant of $f(x,y,z)$ and $g(x,y,z)$ with respect to z, $h(x,y)$, which is curve of (total) degree 8. We will show how to compute the topology of $h(x,y) = 0$ inside $\mathscr{D}$ and, by using the subresultants of $f(x,y,z)$ and $g(x,y,z)$ with respect to z, we analyze how to lift the curve $h(x,y) = 0$ in order to get $\mathscr{T} \cap \mathscr{E}$.

References

[1] L. Gonzalez-Vega, A. Trocado: Tools for analyzing the intersection curve between two quadrics through projection and lifting. Journal of Computational and Applied Mathematics 393 (2021), art. no. 113522.

[2] X.-M. Liu, C.-Y. Liu, J.-H. Yong, J.-C. Paul: Torus/Torus Intersection. Computer-Aided Design and Applications 8 (2011), 465-477.

Bivariate quaternionic factorizations and surfaces that decompose into two circles

Johanna Frischauf[*,1] Niels Lubbes[2] Hans-Peter Schröcker[1]

[1] Department of Basic Sciences in Engineering Sciences, Universität Innsbruck, Technikerstraße 13, 6020 Innsbruck, Austria
johanna.frischauf@uibk.ac.at hans-peter.schroecker@uibk.ac.at

[2]Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences
info@nielslubbes.com

Surfaces that contain two circles through a general point are called *celestial surfaces*. In [3], Skopenkov and Krasauskas characterized celestial surfaces in $\mathbb{R}^3$ up to Möbius equivalence. They showed that a celestial surface in $\mathbb{R}^3$ is Möbius equivalent to either

(I) the stereographic projection of a pointwise product of circles in S^3 or

(II) the pointwise sum of circles in $\mathbb{R}^3$ or

(III) the stereographic projection of a quartic surface in S^3,

where $S^3 \subseteq \mathbb{R}^4$ denotes the 3-dimensional unit sphere. Its elements are identified with unit quaternions. Surfaces of above shape may overlap. Indeed, there exist celestial surfaces which are Möbius equivalent to surfaces of type I and type III. For this reason, we refine above classification result. Our refinement states that a celestial surface in $\mathbb{R}^3$ is, up to Möbius equivalence, either of type I or of type II or

(III)' a smooth cubic surface that contains exactly 2 or 6 circles through each point or

(IV)' a quadratic surface that is neither a sphere nor a one-sheeted circular hyperboloid.

Moreover, we show that a celestial surface in $\mathbb{R}^3$ is not Möbius equivalent to both, surfaces of type I and type II. We relate celestial surfaces in $\mathbb{R}^3$ to bivariate polynomials with quaternion coefficients. We provide a necessary and sufficient condition for bivariate quaternionic polynomials to admit a univariate linear left or right factor and use this algebraic result as an important auxiliary tool for the proofs of the claimed geometric statements.

Figure 1: Celestial surfaces of type I (left) and type II (right).

References

[1] J. Lercher, H.-P. Schröcker: A multiplication technique for the factorization of bivariate quaternionic polynomials, Adv. Appl. Clifford Algebras 32 (2021).

[2] N. Lubbes: Translational and great Darboux cyclides, arXiv:1306.1917 (2022).

[3] M. Skopenkov, R. Krasauskas: Surfaces containing two circles through each point, Math. Ann. 373 (2019), 1299-1327.

Counting graph realizations on the sphere by graph splitting

Matteo Gallet[1] Georg Grasegger[*,2] Niels Lubbes[2] Josef Schicho[3]

[1]University of Trieste

[2]Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences

[3]Johannes Kepler University Linz, Research Institute for Symbolic Computation (RISC)

Minimally rigid graphs have finitely many realizations on the complex sphere, up to rotations, for a generic choice of edge lengths. The three-prism graph for instance generically has 32 complex realizations on the sphere (see Figure 1 for example realizations). This finite number $c^{\circ}(G)$ can be computed by the complex solutions of a system of equations or by a recent combinatorial algorithm [1].

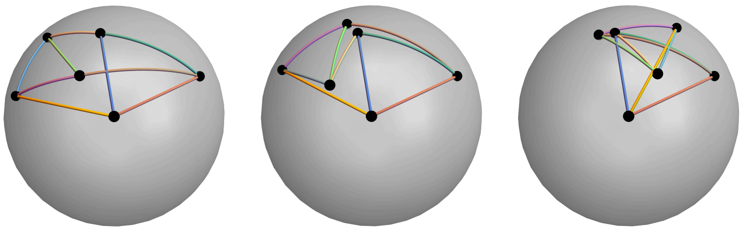


Figure 1: Three non-congruent realizations of the three-prism graph on the sphere.

In this talk we present how the idea from [2] of splitting a minimally rigid graph into two flexible graphs, so called calligraphs, can be used to get a faster recursive algorithm for computing $c^{\circ}(G)$. If a graph G is splittable into A and B we have $c^{\circ}(G) = [A] \cdot [B]$, where $[\cdot]$ denotes the class of a calligraph. This class is related to the curve obtained from tracing a specific vertex in the flexible calligraph. Configurations of a minimally rigid graph can also be interpreted as elements of the moduli space of stable curves with marked points. In this interpretation we show how to recursively compute the class of a calligraph and therefore the number of complex realizations of a minimally rigid graph on the sphere.

References

[1] M. Gallet, G. Grasegger, and J. Schicho. Counting realizations of Laman graphs on the sphere. *Electronic Journal of Combinatorics*, 27(2):P2.5 (1–18), 2020. doi:10.37236/8548.

[2] G. Grasegger, B. El Hilany, and N. Lubbes. Coupler curves of moving graphs and counting realizations of rigid graphs. 2022. Preprint. doi:10.48550/arXiv.2205.02612.

Second order geometric distance fields

Anna Lili Horváth*, Róbert Bán Gábor Valasek

Eötvös Loránd University

We propose a geometric generalization of signed distance fields. While an algebraic distance field stores polynomials that approximate the signed distance function, we store local proxy geometries at each sample.

The order 0 field stores the footpoint. In 2D, the first and second order variants of our construct use tangent lines and osculating circles, respectively. In 3D, the first order realization of our construct encodes an oriented plane. The second order variant uses tori, that reconstruct the footpoint geometric invariants up to the second order, that is, position, surface normal, and principal curvatures and directions.

We show that these provide locally accurate approximations to the signed distance function of the input in first and second order. We consider the problem of filtering the discrete samples to combine them into a continuous approximation beyond tensor product linear interpolation.

References

[1] Bán, Róbert Valasek, Gábor. (2021). Geometric Distance Fields of Plane Curves. Acta Cybernetica. 25. 10.14232/actacyb.289248.

Singularity Distance Computation for Parallel Manipulators of Stewart-Gough Type

Aditya Kapilavai* Georg Nawratil

Institute of Discrete Mathematics and Geometry, TU Wien
Wiedner Hauptstraße 8-10/104, 1040 Wien, Austria
{akapilavai, nawratil}@geometrie.tuwien.ac.at

Under the term parallel manipulators of Stewart-Gough type we summarize mechanisms, where the moving platform is connected to the base by a certain number of prismatic (P) legs according to the robots Degree-of-Freedom (DoF). For planar devices the legs are anchored by passive revolute (R) joints and for spatial ones by passive spherical (S) joints.

In so-called *singular* (also known as *shaky*) *configurations* these manipulators gain at least one instantaneous DoF. Therefore, minor variations in the manipulator geometry (e.g. backlash in passive joints or uncertainties in the actuation of the P joints) can significantly affect the realized configuration. Another phenomenon that also appears close to these configurations is that the prismatic actuator forces can become very large, resulting in a breakdown of the manipulator. Therefore, singular configurations and their vicinity should be avoided. In this context we consider 3-RPR manipulators and present a comparison of singular distances with respect to extrinsic [1] and intrinsic [2] metrics along a 1-parametric motion. Note that different metrics can be used depending on the chosen interpretations of the platform/base; e.g. as triangular plate or as pin-jointed triangular bar structure.

There also exist so-called *architecture singularities* referring to robot designs, which are shaky in every configuration. Clearly, these designs have to be avoided but also their vicinity, as every anchor point can be associated with a space of uncertainties (e.g. tolerances in the passive joints or deviations of the platform/base from the geometric model). In this context we consider linear pentapods (5-SPS manipulators with linear platform) and present an approach to measure the distance of a given design from being architectural singular.

For both kinds of singularities the distances are computed as the global minima of constrained optimization problems. Their critical points are found through a generic computational pipeline that relies on algorithms from symbolic and numerical algebraic geometry implemented in `Maple`, `Bertini` [3] and `Paramotopy` [4]. Note that we do not only obtain the singularity distance but also the corresponding closest singular configuration and architecture singularity, respectively. All presented approaches are demonstrated on the basis of illustrative examples.

References

[1] G. Nawratil, "Singularity Distance for Parallel Manipulators of Stewart Gough Type," Proc. of the 15^{th} IFToMM World Congress on Mechanism and Machine Science, pp. 259–268, 2019.

[2] G. Nawratil, "Snappability and singularity-distance of pin-jointed body-bar frameworks," Mechanism and Machine Theory, vol. (167), pp. 104510, 2022.

[3] D.J. Bates, A.J. Sommese, J.D. Hauenstein, C.W. Wampler, "Numerically solving polynomial systems with Bertini," SIAM, 2013.

[4] D.J. Bates, D.A. Brake, M.E. Niemerg, "Paramotopy: Parameter homotopies in parallel,“ 2018 International Congress on Mathematical Software, pp. 28–35, 2018.

Adaptive isogeometric analysis with C^1 hierarchical splines on analysis-suitable G^1 multi-patch geometries

Cesare Bracco[1] Carlotta Giannelli[1] Mario Kapl[*,2]
Rafael Vázquez[3]

[1]Dipartimento di Matematica e Informatica "U. Dini",
Università degli Studi di Firenze, Florence, Italy
{cesare.bracco|carlotta.giannelli}@unifi.it

[2]Department of Engineering & IT, Carinthia University of Applied Sciences
Europastraße 4, 9524 Villach, Austria
m.kapl@fh-kaernten.at

[3]Institute of Mathematics,
École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland
rafael.vazquez@epfl.ch

We present an adaptive isogeometric method for numerically solving (high-order) partial differential equations over a certain class of bivariate multi-patch geometries, called analysis-suitable G^1 [1]. Our technique is based on the construction of a specific C^1 hierarchical spline space with a suitable basis as well as on the design of a refinement algorithm with linear complexity for the corresponding hierarchical meshes. We also discuss key properties of the constructed C^1 hierarchical spline space and study the proposed refinement algorithm. Finally, the potential of our adaptive method for applications in isogeometric analysis is demonstrated by solving the Poisson and the biharmonic problems over different multi-patch geometries, where for all numerical results the convergence rates indicate optimal order.

References

[1] A. Collin, G. Sangalli, T. Takacs: Analysis-suitable G^1 multi-patch parameterizations for C^1 isogeometric spaces, Computer Aided Geometric Design 47 (2016), 93-113.

Locally based construction of analysis-suitable G^1 multi-patch spline surfaces

Andrea Farahat[1] Mario Kapl[2] Aljaž Kosmač[3] Vito Vitrih[*,3]

[1] Flightkeys GmbH, Vienna, Austria; RICAM, Linz, Austria

[2] Carinthia University of Applied Sciences, Villach, Austria

[3] University of Primorska, Koper, Slovenia

Analysis-suitable G^1 (AS-G^1) multi-patch spline surfaces [1, 2] are particular G^1-smooth multi-patch spline surfaces, which are needed to ensure the construction of C^1-smooth multi-patch spline spaces with optimal polynomial reproduction properties. In [3], a global method to construct AS-G^1 planar multi-patch parameterizations has been developed. In this talk we present a *locally* based approach for the design of AS-G^1 multi-patch spline *surfaces*. The approach is based on a Lagrange multiplier method and generates AS-G^1 multi-patch spline surfaces by approximating a given G^1-smooth but non-AS-G^1 multi-patch surface. Several numerical examples demonstrate the potential of the presented technique for the construction of AS-G^1 multi-patch spline surfaces and show that these surfaces are especially suited for applications in isogeometric analysis by solving the biharmonic problem, a particular fourth order partial differential equation, over them.

References

[1] A. Collin, G. Sangalli, and T. Takacs: Analysis-suitable G^1 multi-patch parametrizations for C^1 isogeometric spaces, Comput. Aided Geom. Des., 47 (2016), 93–113.

[2] A. Farahat, B. Jüttler, M. Kapl, and T. Takacs: Isogeometric analysis with C^1-smooth functions over multi-patch surfaces, Comput. Methods Appl. Mech. Engrg. 403 (2023), 115706.

[3] M. Kapl, G. Sangalli, and T. Takacs: Construction of analysis-suitable G^1 planar multi-patch parameterizations, Comput. Aided Des., 97 (2018), 41–55.

Approximation of Disk B-Spline Curves by Circle Skinning

Kinga Kruppa*[,1] Roland Kunkli[1] Miklós Hoffmann[1,2]

[1]Faculty of Informatics, University of Debrecen
Kassai út 26, 4028 Debrecen, Hungary
kruppa.kinga@inf.unideb.hu
kunkli.roland@inf.unideb.hu

[2]Faculty of Informatics, Eszterházy Károly Catholic University
Leányka utca 4, 3300 Eger, Hungary
hoffmann.miklos@uni-eszterhazy.hu

Disk Bézier and disk B-spline curves have been studied for more than a decade now, but they are still in the interest of researchers in computer aided geometric design [1, 2]. Disk Bézier and their extension to disk B-spline curves provide a method for modelers to construct shapes based on circles while keeping the advantages of the well-known and widely used modeling techniques. In this work, we investigate this area with emphasis on self-intersections of the envelope curves. While we show that assuring envelope curves without self-intersections is challenging, we propose an approximation method utilizing skinning curves to address this issue.

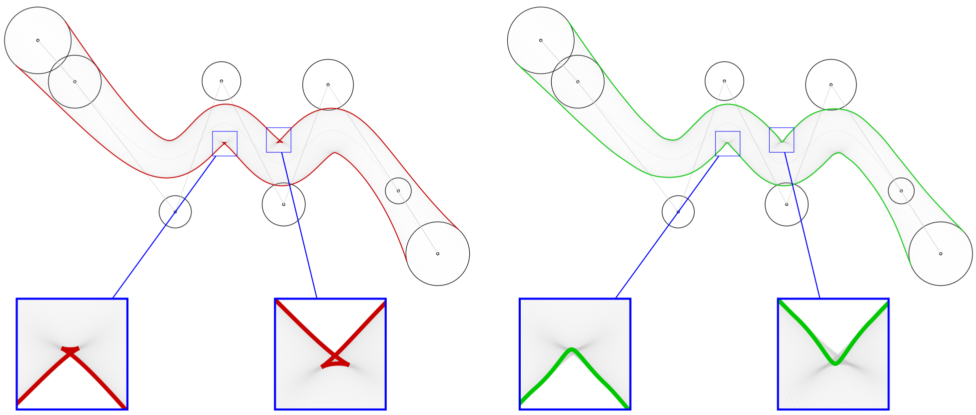


Figure 1: Approximation of the disk B-spline curve (left) with skinning technique (right)

References

[1] X. Ao, Q. Fu, Z. Wu, X. Wang, M. Zhou, Q. Chen, H.S. Seah: An intersection algorithm for disk B-spline curves, Computers & Graphics 70 (2018) 99–107.

[2] Z. Wu, X. Wang, S. Liu, Q. Chen, H.S. Seah, F. Tian: Skeleton-Based Parametric 2-D Region Representation: Disk B-Spline Curves, IEEE Computer Graphics and Applications 41 (2021) 59–70.

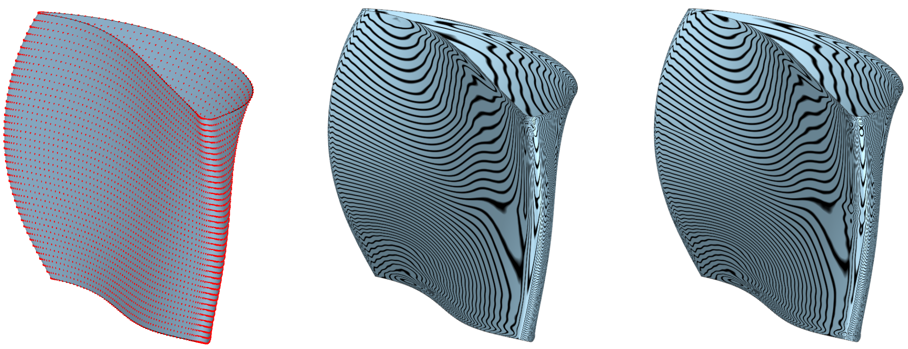
Supported by the ÚNKP-22-4 New National Excellence Program of the Ministry for Culture and Innovation from the source of the National Research, Development and Innovation Fund.

Using Low-Rank Approximations of Gridded Data for Spline Surface Fitting

Dominik Mokriš*,[1] Bert Jüttler[2]

[1] MTU Aero Engines AG
Dachauer Straße 665, 80995, Munich, Germany
dominik.mokris@mtu.de

[2] Institute of Applied Geometry, JKU - Johannes Kepler University Linz
Altenberger Straße 69, 4020 Linz, Austria



The talk describes our recent contribution [1] to the problem of finding bivariate tensor-product spline (or similar) functions that approximate gridded data in the least-squares sense. We propose to apply a low rank approximation of matrices to the data, to find the result by solving a sequence of univariate fitting problems that can be handled efficiently. This can be seen as a generalization of the method proposed by Georgieva and Hofreither [2] that combines cross approximation (which is a particular method for low rank matrix approximation) with spline interpolation.

While the algorithm yields the best least-squares approximation after r steps, where r denotes the rank of the data matrix, terminating it earlier yields a low-rank approximation, which often provides a sufficient level of accuracy. We also present a stopping criterion (based on a lower error estimate) that allows to use the method efficiently in the situation when the required number of degrees of freedom is not known in advance.

Finally, we test the new method on industrial data and discuss its potential for practical applications. We also outline future work, such as the goal of combining the new algorithm with other methods for accelerating surface approximation such as [3].

References

[1] D. Mokriš, B. Jüttler: Using low-rank approximations of gridded data for spline surface fitting, submitted.

[2] I. Georgieva, C. Hofreither: An algorithm for low-rank approximation of bivariate functions using splines, Journal of Computational and Applied Mathematics 310 (2017) 80–91.

[3] S. Merchel, B. Jüttler, D. Mokriš, M. Pan: Fast formation of matrices for least-squares fitting by tensor-product spline surfaces, Computer-Aided Design (2022) article no. 103307.

Interactive design of discrete Voss nets and simulation of their rigid foldings

Martin Kilian Georg Nawratil Matteo Raffaelli*
Arvin Rasoulzadeh Kiumars Sharifmoghaddam*

Institute of Discrete Mathematics and Geometry, TU Wien
Wiedner Hauptstraße 8-10/104, 1040 Wien, Austria
{kilian, nawratil, mraffaelli, rasoulzadeh, ksharif}@geometrie.tuwien.ac.at

Voss nets are surface parametrizations whose parameter lines follow a conjugate network of geodesics. Their discrete counterparts, so called V-hedra, are flexible quadrilateral meshes with planar faces (PQ-meshes), such that opposite angles made by edges around a vertex are equal [1]. This simple geometric condition is highly useful in applications: for instance, when the mesh represents an architectural surface, it ensures that the structure can be realized using initially straight beams and planar panels. As complement to already existing design approaches [2, 3], we developed two interactive tools for designers, architects, and engineers that allow the construction and manipulation of V-hedra in different ways:

(T1) Applying classical optimization techniques, we developed an algorithm for the design of discrete Voss nets and interactive exploration of V-hedra using a handle-based deformation approach. We simulate the flexion of the resulting imperfect V-hedra via a quad soup approach, which also provides the needed joint clearances for rigid folding [4].

(T2) We implemented a V-hedra generator that constructs a modifiable discrete Voss net in a geometrically exact way, from a set of simple conditions already proposed by Sauer [1]. Additionally, the V-hedra generator also incorporates the condition that opposite angles around a vertex are supplementary, which corresponds to an extension of a V-hedral (2×2)-unit to the other side of the parameter line [5]. Our generator can compute and visualize the one-parametric isometric deformation of a discrete Voss net in real time.

The performance and accuracy of (T1) is evaluated by applying the V-hedra generator (T2) to constraints obtained by numerical optimization. In particular we use example surfaces that originate from one-dimensional families of smooth Voss surfaces – each spanned by two isothermal conjugate nets – for which an explicit parametrization is given. This allows us to compare the isometric deformation of the smooth target surface with the rigid folding of the obtained (imperfect) V-hedra.

References

[1] Sauer, R.: Differenzengeometrie. Springer (1970)

[2] Tachi, T.: Freeform rigid-foldable structures using bidirectionally flat-foldable planar quadrilateral mesh. Advances in Architectural Geometry, pp. 87–102 (2010)

[3] Montagne, N., Douthe, C., Tellier, X., Fivet, C., Baverel, O.: Discrete Voss surfaces: Designing geodesic gridshells with planar cladding panels. Automation in Construction **140**:104200 (2022)

[4] Rameau, J.-F., Serre, P., Moinet, M.: Clearance vs. tolerance for mobile overconstrained mechanisms. Mechanism and Machine Theory **136**:284–306 (2019)

[5] Kokotsakis, A.: Über bewegliche Polyeder. Mathematische Annalen **107**:627–647 (1933)

Shape Reconstruction of Profile-affine Surfaces

Arvin Rasoulzadeh* Martin Kilian Georg Nawratil

Institute of Discrete Mathematics and Geometry, TU Wien
Wiedner Hauptstraße 8-10/104, 1040 Wien, Austria
{rasoulzadeh, kilian, nawratil}@geometrie.tuwien.ac.at

A profile-affine surface can be thought of as a generalization of a *surface of revolution* in such a way that the axis of rotation is not fixed at one point but rather traces a smooth path on the base plane. Furthermore, the action, by which the aforementioned surface is obtained does not need to be merely rotation but any "suitable" planar affine transformation applied to the points of a certain smooth profile curve.
The goal of this talk is to reconstruct these surfaces from an already given point cloud (e.g. for the task of reverse engineering [1]). In doing so, a kinematic approach is taken into account, where the algorithm at first finds the direction of the aforementioned axis using the theory of instantaneous equiform motions [2]. Based on planar cuts orthogonal to this direction the algorithm generates a path through which the axis footpoint moves. Finally, by moving all points of the cloud back to a certain plane trough the axis, the planar profile curve is reconstructed.
From an applied point of view, the simple kinematic generation predestines this surfaces class, containing also *translational* and *moulding surfaces* (cf. Fig. 1), for the design process. Moreover, they allow a 1-parametric isometric deformation [3] which can be utilized for transformable designs [4].

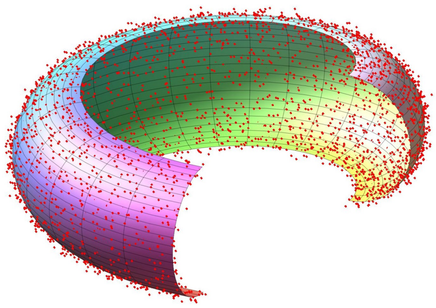


Figure 1: Reconstruction of a *moulding surface* from a given point cloud.

References

[1] Pottmann, H., Hofer, M., Odehnal, B., Wallner, J., 2004. Line geometry for 3D shape understanding and reconstruction. Computer Vision-ECCV 2004 (T. Pajdla, J. Matas eds.), pp. 297–309, Springer.

[2] Odehnal, B., Pottmann, H., Wallner, J., 2006. Equiform kinematics and the geometry of line elements. Beiträge zur Algebra und Geometrie, 47(2), pp.567–582.

[3] Izmestiev, I., Rasoulzadeh, A., Tervooren, J., 2023. Isometric Deformations of Discrete and Smooth T-surfaces, (preprint) arXiv: 2302.08925.

[4] Sharifmoghaddam, K., Nawratil, G., Rasoulzadeh, A., Tervooren, J., 2021. Using Flexible Trapezoidal Quad-Surfaces for Transformable Design. Proc. of IASS Annual Symposia, pp. 3236–3248, IASS.

On a Family of Homogeneous Spaces

Claudiu C. Remsing

Department of Mathematics, Rhodes University,
6140 Makhanda/Grahamstown, South Africa
c.c.remsing@ru.ac.za

We introduce a family $\mathscr{E} = \left(\mathsf{E}^{(p)}, R^2\right)_{p \in R[x]}$ of two-dimensional homogeneous spaces; members of this family are termed *Euclidean-like homogeneous spaces*. A pertinent analysis of a particular sub-family, the homogeneous spaces of the first kind, is undertaken. (The analysis of the homogeneous spaces of the second kind will be treated elsewhere.) Specifically, all the *reductive* members are determined and clearly identified: there are precisely seven types, corresponding to the polynomials $p_1 = x^2, p_2 = (x-1)^2, p_3 = x^2 - 1, p_4 = x(x-1), p_5 = (x-1)(x-\alpha), p_6 = x^2 + 1$, and $p_7 = (x-\beta)^2 + 1$, respectively. (The Lie algebras associated with the principal groups $\mathsf{E}^{(p)}$ were known to Sophus Lie [1]; see, also, [2] and [3], for a modern presentation.)

References

[1] S. Lie: Theorie der Transformationsgruppen, Math. Ann. 16 (1880), 441–528.

[2] A. González-López, N. Kamran and P.J. Olver: Lie algebras of vector fields in the real plane, Proc. London Math. Soc. 64 (1992), 339–368.

[3] B. Komrakov, A. Churyumov and B. Doubrov: Two-Dimensional Homogeneous Spaces, Preprint Series No. 17, University of Oslo, 1993.

Practical methods for Approximate Nearest Neighbor Search on non-Manhattan squares

Minas Dioletis [1,3] Ioannis Z. Emiris[1,3] George Ioannakis[1]
Panagiotis Rigas[1,2,*], Charalambos Tzamos[1,4] Andreas Zamanos[1]

[1] Institute for Language and Speech Processing, Athena Research Center, Greece
{emiris, gioannak, charalambos.tzamos, andreas.zamanos}@athenarc.gr

[2]Electrical & Computer Engineering Dept., Democritus University of Thrace, Greece
panariga@ee.duth.gr

[3]Dept. Informatics & Telecommunication, National & Kapodistrian University of Athens, Greece
{sdi1400272, emiris}@di.uoa.gr

[4]Faculty of Electrical Engineering, Czech Technical University, Prague, Czech Republic
tzamos.charalampos@fel.cvut.cz

The L_∞ metric is popular in applications where Euclidean exactness is not consequential, for its simplicity and therefore computational efficiency [1]. We consider the problem of approximating the L_∞ nearest neighbor (NN) over a set of non-Manhattan, i.e., not axis-aligned squares, to a query point q in the plane. The distance of $q = (x_q, y_q)$ to a square I is defined as $L_\infty(I,q) := \min\{\max(|x - x_q|, |y - y_q|) \mid (x,y) \in I\}$, which is expensive to compute. We define the simpler and faster L^*_∞ metric, applied only to the vertices of I. To calculate the nearest square approximately, we need the error $\varepsilon = L^*_\infty - L_\infty$ to be of small variance. As shown in Fig. 1, we propose an unsupervised clustering method with Kernel Density Estimation (KDE), to decrease the variance of the ε. Our method consists of a clustering based on a Gaussian KDE, a feedback network to optimize the hyperparameters fed to the KDE, and an online search. The error is bounded by $\varepsilon \leq \frac{\sqrt{2}}{4}\ell$, where ℓ is the length of I. In one of our experiments, the accuracy from just using the L^*_∞ without clustering is 85%. For the same experiment, we show that the proposed method can increase the accuracy to 95.1% while having the same complexity. Given the NN, we can find the maximum Manhattan empty square centered at q. We furthermore calculate the approximate speedup of our method when applied to different data structures. Our method is parallelizable by applying a data structure per cluster. The source code may be provided upon request.

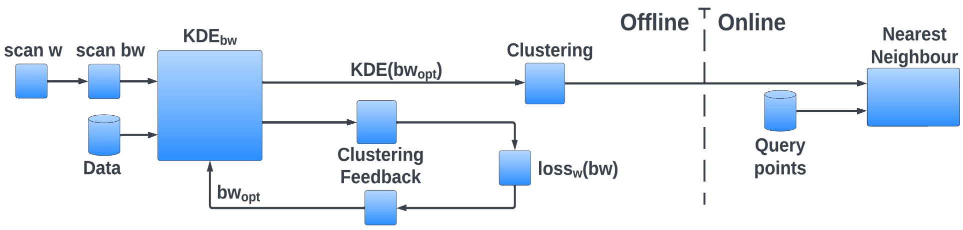


Figure 1: Proposed network topology. The offline part is done once per dataset.

References

[1] Evanthia Papadopoulou and D. T. Lee. 1998. Critical area computation—a new approach. In Proceedings of the 1998 international symposium on Physical design (ISPD '98). Association for Computing Machinery, New York, NY, USA, 89–94. https://doi.org/10.1145/274535.274548

One sided circular arc approximants

Ada Šadl Praprotnik*,[1] Aleš Vavpetič[1,2] Emil Žagar[1,2]

[1] Institute of Mathematics, Physics and Mechanics,
Jadranska 19, 1000 Ljubljana, Slovenia
ada.sadl-praprotnik@imfm.si

[2]University of Ljubljana, Faculty of Mathematics and Physics
ales.vavpetic@fmf.uni-lj.si
emil.zagar@fmf.uni-lj.si

Circular arcs are one of the most fundamental curve shapes used in CAGD. It is well known that they posses a rational parametrization, but no polynomial parametrization exists. In case a polynomial model is required, one must resort to the polynomial approximation. Several optimal approximants of circular arcs by polynomials of different degrees have already been derived, see e.g. [1].

In this talk we consider the problem of optimal one sided circular arc approximants, i.e. polynomials that best approximate a given circular arc from the inside or from the outside. Thus, we are looking for a polynomial $\mathbf{p}$ which interpolates the two boundary points of a circular arc and minimizes the infinity norm of the simplified radial error ψ, i.e,

$$\psi(t) = ||\mathbf{p}(t)||_2^2 - 1, \quad t \in [-1,1],$$

where

$$\mathbf{p}(t) = \sum_{j=0}^{n} B_j^n(t)\mathbf{b}_j$$

is the reparameterized Bernstein-Bézier representation of $\mathbf{p}$. Since we are dealing with one sided approximants, the function ψ must be either non-negative or non-positive on the interval $[-1,1]$. We mainly focus on finding the optimal one sided cubic $\mathscr{G}^0$ approximant of a circular arc with its inner angle $\leq \pi$, but we also consider several other cases, depending on the geometric continuity and the degree of polynomial $\mathbf{p}$.

References

[1] A. Vavpetič: Optimal parametric interpolants of circular arcs, Computer Aided Geometric Design 80 (2020), 101891, 9.

Camera Calibration using Squares, Spheres, or Surfaces of Revolution

Niels Lubbes[1] and Josef Schicho[*,2]

[1] RICAM
Altenberger Straße 69, 4020 Linz, Austria
Niels.Lubbes@ricam.oeaw.ac.at

[2] RISC/JKU and RICAM
Altenberger Straße 69, 4020 Linz, Austria
josef.schicho@jku.at

A camera generates a 2D image of a 3D scenario by a central projection π. A calibrated camera adds a metric to the image plane, so that the distance between two image points p, q is equal to the viewing angle between $\pi^{-1}(p)$ and $\pi^{-1}(q)$. This distance makes the image plane a model of elliptic geometry. Similar to the circular points at infinity in the Euclidean plane, elliptic distance can be defined by a conic in the image plane without real points, called the *elliptic absolute conic*.

When the 3D scenario contains special objects, then the camera can be calibrated from the images of these objects by geometric constructions. The elliptic absolute conic can be constructed from the images of three squares – see [1, Example 8.18] – or from the images of three spheres – see [2]. After reviewing these constructions, we give another reconstruction from three surfaces of revolution. The objects may be combined: we can also calibrate from one square, one sphere, and one object of revolution (see Figure 1).

Figure 1: a scenario that allows calibration – it has squares, spheres, and surfaces of revolution

References

[1] M. Agrawal and L. Davis, Camera calibration using spheres: a semidefinite programming approach, Proc. ICCV (2003), IEEE, 2003.

[2] R. Hartley and A. Zisserman: Multiple view geometry in computer vision. CUP, 2004.

Deep learning based methods for geometric modeling

Felix Scholz*,[1]

[1] Institute of Applied Geometry, JKU - Johannes Kepler University Linz
Altenberger Straße 69, 4020 Linz, Austria
felix.scholz@jku.at

In this talk, we will present different recently developed methods that employ deep neural networks for problems in geometric modeling. We will first consider the problem of reconstructing a smooth geometry from point cloud data. A crucial step for this task is to find a suitable parameterization of the point cloud. The case of parameterizing data sampled from a curve for fitting polynomial curves was considered in [1]. In recent works, we investigated the generalization to the parameterization of 3D point clouds sampled from surfaces using graph convolutional neural networks. To this end, we propose a new graph convolutional layer that takes into account given boundary conditions and propagates them into the interior of the point cloud.

Moreover, we will discuss ongoing research on optimizing isogeometric function spaces for approximating functions with singularities using deep neural networks. Our goal is to improve the accuracy of isogeometric approximations to singular solutions of partial differential equations without increasing the number of degrees of freedom. In our proposed method, we adjust the control points of the underlying spline parameterization of the computational domain based on the output of a trained deep neural network.

Joint work with: Carlotta Gianelli, Sofia Imperatore, Bert Jüttler, Angelos Mantzaflaris, Dany Rios, Thomas Takacs

References

[1] F. Scholz, B. Jüttler: Parameterization for polynomial curve approximation via residual deep neural networks, Computer Aided Geometric Design 85 (2021), 101977.

A Geometric Approach on the Factorization of Motion Polynomials

Zijia Li[1] Hans-Peter Schröcker[2] Johannes Siegele[*,2]

[1] KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences
lizijia@amss.ac.cn

[2] Department of Basic Sciences in Engineering, Universität Innsbruck
{hans-peter.schroecker, johannes.siegele}@uibk.ac.at

Clifford algebras are a powerful tool for the representation of orthogonal transformations in quadratic spaces. Hamiltonian, dual and split quaternions for example can be used to discribe displacements in spherical, Euclidean and hyperbolic geometry, respectively. Rational motions in these spaces can be parametrized by polynomials with coefficients in the corresponding Clifford algebras. Factorization of such motion polynomials corresponds to the decomposition of the parametrized motion into a concatination of elementary submotions like rotations and translations. This in turn can be used to construct mechanisms consisting of joints which can perform these elementary motions.

Hamiltonian quaternion polynomials always admit a factorization which can be computed by a rather simple factorization algorithm. However, both determining factorizability as well as computing factorizations of split or dual quaternion polynomials can be more intricate [1, 2]. In our work, we propose a geometric approach for the construction of a linear left factor of a given motion polynomial. This is done by investigating the intersections of the curve parametrized by the motion polynomial with a quadric containing all points, that represent singular displacements, the so called null quadric. These singular displacements, in general, map the whole space to one unique isotropic image point. We will give conditions on these image points for the existence of a linear left factor and give a method for its construction using these points.

This approach seems beneficial for the computation of a factor in certain exceptional cases, where it circumvents the necessity to solve a system of quadratic equations. It also gives better geometric insight on the multiplication technique for split quaternions in [1] and allows for the generalization of this technique to the Clifford algebra $\mathrm{Cl}_{4,1}$ which is used for conformal kinematics.

References

[1] D.F. Scharler, H.-P. Schröcker. An Algorithm for the Factorization of Split Quaternion Polynomials. Adv. Appl. Clifford Algebras 31, 29 (2021).

[2] Z. Li, H.-P. Schröcker, D.F. Scharler. A Complete Characterization of Bounded Motion Polynomials Admitting a Factorization with Linear Factors. arXiv (2022).

Apollonian de Casteljau–type Algorithms for Complex Rational Bézier Curves

Zbyněk Šír*,[1] Bert Jüttler[2] Josef Schicho[3]

[1] Faculty of Mathematics and Physics, Charles University
Sokolovská 83, Prague 180 00, Czech Republic
zbynek.sir@mff.cuni.cz

[2] Institute of Applied Geometry, Johannes Kepler University
Altenberger Strasse 69, Linz 4040, Austria
bert.juettler@jku.at

[3] RISC, Johannes Kepler University
Altenberger Strasse 69, Linz 4040, Austria
josef.schicho@risc.jku.at

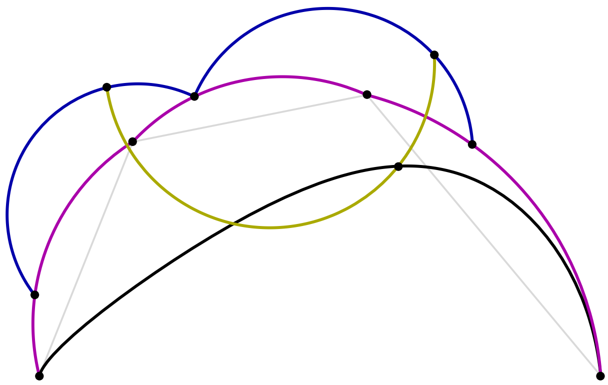


Figure 1: Sample figure

After reviewing the classical de Casteljau algorithms for polynomial and rational Bézier curves we present two extensions. The first one consists in using one root of the denominator for the construction of all the points in one level of the de Casteljau algorithm, [2]. Consequently, for a rational curve of degree n there are $n!$ distinct algorithms depending on the ordering of the denominator roots.

The second extension consists in considering the de Casteljau algorithm over the complex numbers [1]. The resulting curves have generically an even degree and exhibit the maximal circularity. Segments in the algorithm are replaced by circular arcs. The two extensions can be combined producing an Apollonian algorithm in which the bipolar coordinates play a key role.

References

[1] Sánchez-Reyes, J., Complex rational Bézier curves. Computer Aided Geometric Design 26, 865-876, 2009.

[2] Šír, Z., Jüttler B.: On de Casteljau-type algorithms for rational Béezier curves. Journal of Computational and Applied Mathematics 288, 244-250, 2015.

On orthogonal trajectories and zeroes of curvature of isoptics
The special case: isoptics of Fermat curves

Magdalena Skrzypiec

Maria Curie-Skłodowska University, Poland
magdalena.skrzypiec@umcs.pl

What are the orthogonal trajectories of the isoptics of a given convex curve? We will present a Cauchy problem, the solution of which produces the parametric form of the orthogonal trajectories we seek out. We consider the existence and uniqueness of the solution of this problem for isoptics of ovals.

Do shapes of those trajectories depend on the convexity of the isoptics of a given curve? We will examine these dependencies.

The isoptics of the Fermat curve $x^n + y^n = 1$, for n-even, described in [2], have not yet been thoroughly investigated. The Fermat curves are not ovals, and their curvature vanishes at isolated points, so we have to be careful studying their isoptics. We will examine the existence of their limit angle, that is, the angle at which convexity of these curves is lost, see [3]. We also consider their orthogonal trajectories.

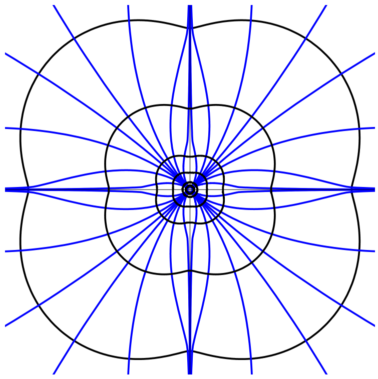


Figure 1: Isoptics of the curve $x^4 + y^4 = 1$ and their orthogonal trajectories

Joint work with Aharon Naiman (Jerusalem College of Technology, Israel), Witold Mozgawa (Academy of Zamość, Institute of Social and Economic Sciences, Poland) and Piotr Pikuta (Maria Curie-Skłodowska University, Poland).

References

[1] Waldemar Cieślak, Andrzej Miernowski, and Witold Mozgawa. Isoptics of a closed strictly convex curve. *Global differential geometry and global analysis*, 1991, (Berlin, 1990), 28-35, *Lecture Notes in Math.*, **1481**, Springer.

[2] Thierry Dana-Picard, Aharon Naiman, Witold Mozgawa, and Waldemar Cieślak. Exploring the isoptics of Fermat curves in the affine plane using DGS and CAS. *Math. Comput. Sci.*, **14**, No. 1, 45-67 (2020).

[3] Magdalena Skrzypiec. Convexity limit angles for isoptics *Beiträge zur Algebra und Geometrie*, **63**, 55-67, (2021).

On the Singularities of the General 6-SPS platform

Christoforos Spartalis*,[1] Jose Capco[2]

Unit Geometry and Surveying, University of Innsbruck
Technikerstr. 13, 6020 Innsbruck, Austria
[1]christoforos.spartalis@uibk.ac.at

[2]jose.capco@uibk.ac.at

We will present our results on the investigation of the singularities of the general 6-SPS platform. This is a variety Sing $\subset \mathrm{SE}(3)$ and our goal is to give a description of its topology by extending a similar approach that was followed on its planar counterpart, i.e. the 3-RPR platform [3]. In this direction, we aim to analyse the topology of every affine cubic surface $Sing_R$ that arises after fixing the orientation of the platform and deduce results for Sing in total. In [1], Coste and Moussa, actually prove that the generic fiber Σ of the family of these surfaces parametrized by $\mathrm{SO}(3)$, i.e. an affine smooth cubic surface over $\mathbb{R}(\mathrm{SO}(3))$, is rational over $\mathbb{R}$ by proving that is in birational equivalence with a quadric hypersurface in the projective space over $\mathbb{R}(\mathrm{SO}(3))$. Additionally, recent work by Finashin and Kharlamov [2] classifies the smooth real affine cubic surfaces that are transveral at infinity and describe the wall-crossings between such classes. We are going to discuss how the results from these two state of the art works ([1, 2]) can influence the research being carried out on the singularities of the general 6-SPS platform and how they could potentially be used to tackle the open problem of the determination of the number of connected components in the singularity-free space of such mechanisms.

References

[1] **M. Coste, S. Moussa**, *Rationality of the Locus of Singularities of the General Gough-Stewart Platform*, SIAM Journal on Applied Algebra and Geometry, Vol. 4, No. 3, pp. 401–421 (2020). Online: `https://doi.org/10.1137/19M1253277`.

[2] **S. Finashin, V. Kharlamov**, *On Affine Real Cubic Surfaces*.
Online: `https://arxiv.org/abs/2208.09502`, Accessed: 02.2023.

[3] **C. Spartalis, J. Capco**, *Topology of the singularities of 3-RPR planar parallel robots*, Computer Aided Geometric Design, Vol. 99, p. 102150 (2022). Online: `https://doi.org/10.1016/j.cagd.2022.102150`.

Smooth and Discrete Cone-nets

Jonas Tervooren*,[1] Christian Müller[2] Martin Kilian[3]

Institute of Discrete Mathematics and Geometry, TU Wien
Wiedner Hauptstrasse 8-10/104 A-1040 Vienna, Austria

[1]jtervooren@geometrie.tuwien.ac.at

[2] cmueller@geometrie.tuwien.ac.at

[3] kilian@geometrie.tuwien.ac.at

Cone-nets are conjugate nets on a surface such that along each individual curve of one family of parameter curves there is a cone in tangential contact with the surface. The corresponding conjugate curve network is projectively invariant and is characterized by the existence of particular transformations. We study properties of that transformation theory and illustrate how several known surface classes appear within our framework. We present cone-nets in the classical smooth setting of differential geometry as well as in the context of a consistent discretization with counterparts to all relevant statements and notions of the smooth setting. Special emphasis deserve smooth and discrete tractrix surfaces as those cone-nets which are characterized as principal nets with constant geodesic curvature along one family of parameter curves. For more information see [1]. In the talk, we will give a general introduction to cone-nets and then present the new results and constructions for both smooth and discrete surfaces.

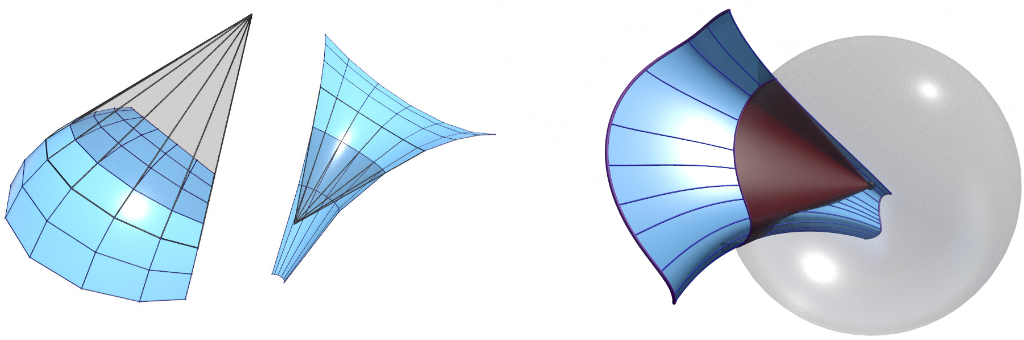


Figure 1: Left: Two discrete cone-nets, that are related by a conical Combescure transformation. Right: A smooth principle cone-net, the parameter lines on the cones are spherical and have constant geodesic curvature.

References

[1] C. Müller, M. Kilian, J. Tervooren: Smooth and Discrete Cone-nets, Results in Mathematics (RIMA), accepted February 2023, preprint: https://www.geometrie.tuwien.ac.at/geom/ig/publications/cone-nets/cone-nets.pdf.

Higher-Order Geometric Hermite Interpolation of Surfaces

Gábor Valasek

Shapr3D, Hungary

We present a characterization of geometric Hermite interpolation of surfaces beyond order two. We consider parametrization by lines of curvatures as the surface analogue of arc-length or natural parametrization of curves. By expressing arbitrary patches as reparametrizations of these natural surface parametrizations, we derive a recurrence formula for the Darboux coordinates of the higher-order partial derivatives of arbitrary surfaces. This formulation explicitly separates the degrees of freedom of parametrization from the constraints of geometric Hermite interpolation, simplifying numerical parametrization optimizations.

The geometric data to be interpolated are expressed as the projections of the arc-length derivatives of the lines of curvatures onto the unit surface normal, which, in turn, are governed by the local differential geometry of the lines of curvatures. In extension, the signed distance function is considered as the implicit realization of the natural form of its boundary, since its partial derivatives only depend on the local differential geometry at and the distance to the footpoint.

References

[1] X. Song, B. Jüttler and A. Poteaux, Hierarchical Spline Approximation of the Signed Distance Function, 2010 Shape Modeling International Conference, Aix-en-Provence, France (2010), 241-245.

[2] Han Kyul Joo and Tatsuya Yazaki and Masahito Takezawa and Takashi Maekawa: Differential geometry properties of lines of curvature of parametric surfaces and their visualization, Graphical Models, Volume 76, Number 4 (2014), 224-238.

An Improved Algorithm for Scattered Data Interpolation using Quartic Triangular Bézier Surfaces

Krassimira Vlachkova*,[1] Krum Radev[1]

[1] Sofia University "St. Kliment Ohridski"
Faculty of Mathematics and Informatics
1164 Sofia, Bulgaria
krassivl@fmi.uni-sofia.bg
kvradev@uni-sofia.bg

We revisit the problem of interpolation of scattered data in $\mathbb{R}^3$ and propose a solution based on Nielson's minimum norm network and triangular Bézier patches. We aimed at solving the problem using the least number of polynomial patches of the smallest possible degree. We propose an alternative to the previously known algorithms, see (Clough and Tocher [1]) and (Shirman and Séquin [2, 3]). Although conceptually similar, our algorithm differs from the previous in all its steps. As a result the complexity of the resulting surface is reduced and its smoothness is improved. We present results of our numerical experiments.

Acknowledgments. This work was supported in part by Sofia University Science Fund Grant (2023), and European Regional Development Fund and the Operational Program "Science and Education for Smart Growth" under contract No BG05M2OP001-1.001-0004 UNITe.

References

[1] R.W. Clough, J.L. Tocher, Finite elements stiffness matrices for analysis of plate bending, *Proceedings of the 1st Conference on Matrix Methods in Structural Mechanics* 66-80 (1965), 515-545.

[2] L.A. Shirman, C.H. Séquin, Local surface interpolation with Bézier patches, Comput. Aided Geom. Des. 4 (1987), 279-295.

[3] L.A. Shirman, C.H. Séquin, Local surface interpolation with Bézier patches: errata and improvements, Comput. Aided Geom. Des. 8 (1991), 217-221.