Implicitization and Distance Bounds

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Summary. We address the following problem: given a curve in parametric form, compute the implicit representation of another one that approximates the parametric curve on a certain domain of interest. We study this problem from the numerical point of view: what happens with the output curve if the input curve is slightly changed? It is shown that for any approximate parameterization of the given curve, the curve obtained by an approximate implicitization with a given precision is contained within a certain perturbation region.

1 Introduction

In geometric modeling and computer aided design, various different representations for curves and surfaces exist, such as implicitly defined curves and surfaces, parametric representations by (piecewise) rational functions, procedurally defined surfaces, or triangular meshes. The duality of implicit and parametric representations makes each of them especially well suited for certain applications.

In order to exploit the potential benefits of both representations, one has to be able to transform one representation into the other. Theoretically, the conversion from an implicit representation to a parametric one (implicitization) is always possible, whereas the reverse conversion (parameterization) is generally impossible. This paper focuses on implicitization.

Various symbolic-computation-based methods for implicitization have been introduced, based on Gröbner bases [AGR95, Buc88], resultants [Bus01, MM02], moving curves and surfaces [ZSCC03, SSQK94], residue calculus [EM04] or on other methods [GV97, CD04]. These techniques produce an exact implicit representation.

In applications, the input curve (or surface) is often not given exactly, but it may be contaminated by numerical errors. In this situation, using approximate techniques may be more appropriate [Che03, CGKW01, Dok01, DT03, SJS04]. Also, these techniques are able to deal with the more complicated data needed for industrial applications [S+0x].

This paper studies the effects caused by using an approximate implicitization. More precisely, we address the following problem: Given a parametric representation $\mathbf{p} = \mathbf{p}(u)$, $u \in [0, 1]$, of a planar curve segment with domain $[0, 1]$. Let $\mathcal{V}(\mathbf{p}) = \mathbf{p}([0, 1])$ be the point set defined by the curve. Consider the zero set $\mathcal{C}_f$ of an approximate implicitization f of the parametric curve, where the coefficients of the residuum $f \circ \mathbf{p}$ are bounded by a positive constant ϵ. How close is it to the given curve? As an answer, we derive an upper bound for the one-sided Hausdorff distance of $\mathcal{V}(\mathbf{p})$ and $\mathcal{C}_f$. The bound is valid for all approximate implicitizations f, where the residuum can be bounded by ϵ.

Our approach is based on earlier results of [SS04], who introduced a condition number that allows to estimate the distance of the two coefficient vectors of two approximate implicitizations. For curves (and similarly surfaces) with a high condition number, the computation of the coefficients of an approximate implicitization is not numerically stable, no matter which numerical method for implicitization is chosen.

Unfortunately, even if the coefficients can be computed in a numerically stable way, it is not guaranteed that the zero sets of two approximate implicitizations are close in a geometric sense. However, one can estimate the one-sided Hausdorff distance of zero sets of an exact and a perturbed equation, using a result by [AJK04]. This leads to a constant expressing the robustness of an implicit representation.

Combining these two robustness results allows to examine the suitability of a given rational parametric curve for approximate implicitization. A curve could be said to be "well behaved", if

(1) the computation of the coefficients of an approximate implicitization is numerically stable, and
(2) the resulting implicit representation is geometrically robust with respect to small perturbations of its coefficients.

For the sake of simplicity, the results in this paper are presented in the case of planar curves. In principle, they can be generalized to hypersurfaces in any dimension (such surfaces in three-dimensional space), but the computations are especially simple in the planar situation.

The paper is organized as follows. After introducing some notations in the next section, we consider the stability of the implicitization in Section 3. In Section 4 we give a distance bound between a parametrically and an implicitly given curve. Section 5 collects the obtained results which enable us to show, that for any approximate parameterization $\mathbf{p}_\delta$ of $\mathbf{p}$, the curve obtained by any approximate implicitization with precision ϵ lies within a certain perturbation region. Finally we conclude the paper.

2 Preliminaries

Throughout this paper, we shall use three spaces P, I, and R of polynomials, where the letters stand for parameterization, implicitization, and residuals respectively.

Consider two positive integers n, d, and let P be the set of *triples of polynomials* of degree less or equal than n in the variable u over $\mathbb{R}$. These polynomials will be described by their coefficient vectors with respect to the Bernstein basis with respect to the parameter domain $[0, 1]$.

The elements of P define rational parametric curves of degree less or equal than d in homogeneous coordinates,

$$\mathbf{p}(u) := (p_0(u), p_1(u), p_2(u)) = \sum_{i=0}^{n} \mathbf{b}_i \binom{n}{i} u^i (1-u)^{n-i}, \quad u \in [0, 1], \tag{1}$$

where the corresponding Cartesian coordinates are $(\frac{p_1}{p_0}, \frac{p_2}{p_0})$.

Let I be the set of all *homogeneous polynomials* of degree d in the variables x_0, x_1, x_2 over $\mathbb{R}$. These functions serve to represent a (possibly approximate) implicitization of a given curve $\mathbf{p}$, by an algebraic curve segment $\mathcal{C}$ of order d

$$\mathcal{C} = \{\, (x_1, x_2) \,|\, (x_1, x_2) \in \Omega \subset \mathbb{R}^2 \wedge f(1, x_1, x_2) = 0\}. \tag{2}$$

Such an algebraic curve is defined in a certain planar domain $\Omega \subset \mathbb{R}^2$ by the zero contour of a bivariate polynomial $f \in I$ of degree d. This polynomial is given by its homogeneous monomial representation

$$f(x_0, x_1, x_2) := \sum_{i,j,k \in \mathbb{N},\; i+j+k=d} b_{ijk}\, x_0^i x_1^j x_2^k \tag{3}$$

with certain coefficients b_{ijk}. Sometimes we will also use the inhomogeneous representation $f(1, x_1, x_2)$. If no confusion can arise, we shall write $f(x_1, x_2)$ instead. Alternatively, one can use a Bernstein-Bézier representation with respect to a suitable domain triangle, see [FHK02].

Finally, we denote by R the set of *polynomials* of degree less or equal than nd in the variable u over $\mathbb{R}$. Again, these polynomials will be described by their coefficient vectors with respect to the Bernstein basis with respect to the parameter domain $[0, 1]$.

Clearly, the sets P, I, R are linear spaces with a finite dimension, which can be identified with $\mathbb{R}^m$, where m is the corresponding dimension. Then, the usual inner product in $\mathbb{R}^m$ defines an inner product in P, I, and R respectively. The associated norm is defined by $\|x\| := \sqrt{\langle x, x \rangle}$. In order to simplify the notation, any element of R and I will be identified with its coefficient vector with respect to the corresponding basis.

Finally, we define the *evaluation map*

$$\mathrm{eval} : I \times P \to R \;\text{ by }\; (f, \mathbf{p}) \mapsto \mathrm{eval}(f, \mathbf{p}) = f \circ \mathbf{p}.$$

Note that the evaluation map is linear in its first argument, but non–linear in the second one.

3 Condition number of the implicitization

Following earlier results of [SS04], we define the condition number of the curve implicitization problem, and we give an algorithm for computing it.

3.1 Definition and computation of the condition number

Assume that $\mathbf{p} \in P$, $\|\mathbf{p}\| = 1$, $f \in I$, and $f \circ \mathbf{p} = 0$.

For any $h \in I$, $\mathbf{p} \in P$ we get

$$\mathrm{eval}(h, \mathbf{p}) = M_{\mathbf{p}} \cdot h \tag{4}$$

where $M_{\mathbf{p}}$ is a matrix depending on the coefficients of $\mathbf{p}$ of size $\bar{d} \times \bar{n}$, where

$$\bar{d} = \binom{dn + 2 - 1}{dn}, \quad \text{and} \quad \bar{n} = \binom{d+2}{d}. \tag{5}$$

Using a singular value decomposition, the matrix $M_{\mathbf{p}}$ can be factorized as

$$U \cdot \Sigma \cdot V^t, \tag{6}$$

where $\Sigma \in \mathbb{R}^{\bar{d} \times \bar{n}}$ is a diagonal matrix containing the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\bar{n}} \geq 0$, and $U \in \mathbb{R}^{\bar{d} \times \bar{d}}$, $V \in \mathbb{R}^{\bar{n} \times \bar{n}}$ are orthogonal matrices. We have the following result:

Proposition 1. *Let f and $\mathbf{p}$ as above, and assume additionally that $\|f\| = 1$. Then the following are true.*

1. *The smallest singular value of $M_{\mathbf{p}}$ is zero.*
2. *The coefficient vector of the polynomial f is a right singular vector to the smallest singular value $\sigma_{\bar{n}} = 0$.*
3. *The right singular vector belonging to the least but one singular value $\sigma_{\bar{n}-1}$, where $\bar{n} = \binom{d+2}{d}$, minimizes the function $g \mapsto ||\mathrm{eval}(g, \mathbf{p})||$ in the unit sphere of $f^{\perp}$, where $f^{\perp} := \{J \in I | \langle f, J \rangle_I = 0\}$.*

If $\sigma_{\bar{n}-1} = 0$, then there are several linearly independent equations h with $\mathrm{eval}(h, \mathbf{p}) = 0$. If $\sigma_{\bar{n}-1}$ is small, we are close to such a case. Hence, in some sense, the reciprocal of $\sigma_{\bar{n}-1}$ is a numerical measurement of the uniqueness of the implicitization.

Let us now drop the assumption on $\mathbf{p}$ that there exist f such that $f \circ \mathbf{p} = 0$. We still keep the assumption $\|\mathbf{p}\| = 1$. We define the *condition number* K as the inverse of the *formally second smallest* singular value of $M_{\mathbf{p}}$,

$$K = \frac{1}{\sigma_{\bar{n}-1}}. \tag{7}$$

With “formally second smallest singular value”, we mean that we take multiplicities into account. For instance, if 0 is a multiple singular value, then the condition number is infinity.

Note, that the condition number K depends not just on $\mathbf{p}$, but also on the (estimated) degree d of f. If $\|\mathbf{p}\| \neq 1$, then the condition number always refers the condition number of the normed equation.

Remark 1. In order to compute K, the implicit equation of the parametrically given curve is not needed. The computation of the formally second smallest singular value is easier than the computation of the implicit equation, at least numerically. Singular values are numerically stable, whereas the implicitization problem can be very badly conditioned.

Remark 2. If the last singular value of $M_{\mathbf{p}}$ is sufficiently small, then there exist an f of degree d such that $\mathrm{eval}(f, \mathbf{p})$ is small. Namely, f is the right singular vector belonging to the smallest singular value.

Here is an algorithm for computing the condition number.

Algorithm 1 ("Condition Number")
Input: A triple of polynomials $\mathbf{p} = (p_0, p_1, p_2)$ of total degree n in the parameter u, such that $\|\mathbf{p}\| = 1$, and an $d \in \mathbb{Z}$.
Output: Condition number K of the implicitization problem.

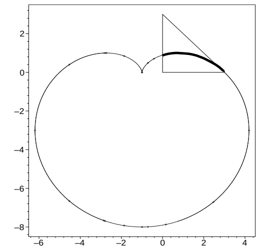
1. Initialize $M_{\mathbf{p}}$ by a zero matrix.
   for each b_i in the basis B_I of I, $i = 1, \ldots, \bar{n}$
   a) substitute $\mathbf{p}$ into b_i,
   b) expand the result in the basis B_R of R
   c) append the column to $M_{\mathbf{p}}$
   (Now we have constructed the matrix $M_{\mathbf{p}}$)
2. Compute the singular value decomposition of the matrix $M_{\mathbf{p}}$.
3. The condition number is $K = 1/\sigma_{\bar{n}-1}$, where $\bar{n} = \binom{d+2}{d}$

3.2 Examples

The planar algebraic curves shown in Table 1 will serve as test examples. We considered segments of four well known curves. First we computed an approximate parametric representation and then an approximative implicitization. Both representations are given in the Table. In addition, the domain triangle has been specified, and it is also shown in the figures. Although the computations are done in Bernstein-Bézier-representation, the parametric representation is given with respect to the monomial basis, since many coefficients vanish and a basis transformation is relatively simple [Far91].

Using Algorithm 1 above we compute the condition number of the four curves. The condition number of the implicitization problem depends not only on the parametric form, but also on the estimate of the degree of the implicit form. From classical algebraic geometry it is known that any degree n polynomial or rational parametric curve can be represented using a degree n algebraic equation. However, if the implicitization is only done approximatively, a lower degree than n may be sufficient in order to gain a result of a desired accuracy.

The Cardioid

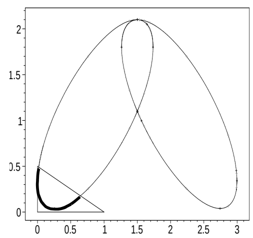


$$\begin{pmatrix} -\frac{23134208}{5}u+4194304+\frac{63799808}{25}u^2-\frac{87973954}{125}u^3+\frac{15527402881}{160000}u^4 \\ -23134208u+12582912+\frac{63799808}{5}u^2-\frac{263921862}{125}u^3-\frac{15527402881}{160000}u^4 \\ \frac{46268416}{5}u-\frac{63799808}{5}u^2+\frac{703791632}{125}u^3-\frac{15527402881}{20000}u^4 \end{pmatrix}$$

$$0.00775+0.37537x_1+0.19263x_2+0.00762x_1^2+0.061820x_1x_2-0.41813x_2^2 +0.03663x_1^3+0.35689x_1^2x_2+0.42381x_1x_2^2-0.68875x_2^3$$

$(0,0),(3,0),(0,3)$

The Tacnode

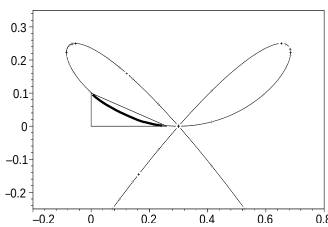


$$\begin{pmatrix} 0.95233\cdot 10^{-2}u^3+0.43723u^2-0.5478u+0.89 \\ 0.15\cdot 10^{-1}-0.1848u+0.1132\cdot 10^{-1}u^3+0.6836u^2+0.8523\cdot 10^{-4}u^4 \\ 0.11043\cdot 10^{-3}u^4+0.20358\cdot 10^{-1}u^3+0.93016u^2-1.2295u+0.429 \end{pmatrix}10^{-2}$$

$$0.5449-0.64792x_1-0.25435x_2+0.3853x_1^2-0.26083x_1x_2+0.04559x_2^2$$

$(0,0),(1,0),(0,0.5)$

The Bicorn

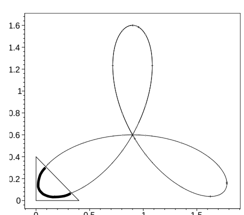


$$\begin{pmatrix} 1 \\ 0.399+0.5335u-0.16637u^3-0.0.9075u^2 \\ 0.28435u^2+0.1078u-0.0915u^4-.0.06655u^3+0.0099 \end{pmatrix}$$

$$-0.35901-0.34195x_1+0.33064x_2-0.22577x_1^2+ 0.57652x_1x_2-0.10127x_2^2-0.11083x_1^3+0.46583x_1^2x_2 -0.14776x_1x_2^2+0.01032x_2^3$$

$(0,0),(0.25,0),(0,0.1)$

The Trifolium



$$\begin{pmatrix} 1.86624u^4+3.59424u^3+4.58784u^2-1.86336u+0.19629 \\ -1.34784u^2-1.08864u+0.45096+3.31776u^4+3.31776u^3 \\ 1.1881+3.6576u^2+1.5696u+2.0736u^4+2.0736u^3; \end{pmatrix}$$

$$-0.1034x_1^4+0.37226x_1^3-0.020681x_1^2x_2^2-0.62044x_1^2x_2 -0.39088x_1^2-0.37226x_1x_2^2+0.11168x_1x_2+0.10051x_1-0.1034x_2^4 +0.35158x_2^3-0.57701x_2^2+0.15076x_2-0.01312$$

$(0,0),(0.4,0),(0,0.4)$

Table 1. The four examples. From top to bottom, the table shows name, approximative parametric representation, approximative implicit representation and domain triangle.

curve	degree	$\sigma_{\bar{n}}$	K
cardioid	3	$0.22976 \cdot 10^{-5}$	$0.43523 \cdot 10^{6}$
tacnode	2	0.07907	12.687
trifolium	4	$0.40345 \cdot 10^{-3}$	2478.62
bicorn	4	$0.50253 \cdot 10^{-8}$	$0.19899 \cdot 10^{9}$

Table 2. Condition numbers of the implicitization.

Example 1. Table 2 contains the singular value $\sigma_{\bar{n}}$, the condition number K and the degree of the implicit representation of each curve. In the case of the tacnode and the trifolium we can say that the implicitization problem is well-conditioned.

4 Distance between implicit and parametric curves

This Section is dedicated to the computation of the distance between a parametric and an implicitly given curve. For quantifying the distance between the two curves we use the concept of the Hausdorff distance.

Since we use a triangular Bernstein-Bézier-representation, the implicit curve is tied to its domain triangle. In the remainder of this paper we assume that this triangle is chosen in such way, that the whole parametric curve is contained in it.

In order to avoid some technical difficulties which may arise if the parametric curve hits the boundary of the triangular domain $\triangle$, we consider the distance between $\mathcal{V}(\mathbf{p})$ and

$$\mathcal{C}^* := \mathcal{C} \cup \partial\triangle. \tag{8}$$

More precisely, we consider the distance

$$\mathrm{HD}_{\triangle}(\mathcal{V}(\mathbf{p}), \mathcal{C}^*) = \sup_{\mathbf{y} \in \mathcal{V}(\mathbf{p}) \cap \triangle} \; \inf_{\mathbf{x} \in \mathcal{C}^* \cap \triangle} ||\mathbf{x} - \mathbf{y}||. \tag{9}$$

We call this distance the *one–sided Hausdorff distance*[4] of $\mathcal{V}(\mathbf{p})$ and $\mathcal{C}^*$ with respect to the domain triangle $\triangle$. For the effect of replacing $\mathcal{C}$ by $\mathcal{C}^*$, see Figure 3.

Lemma 1. *Let $\mathcal{C} \subset \triangle \subset \mathbb{R}^2$ be an algebraic curve which is defined by the homogeneous polynomial f of degree d and $\mathcal{C}^* := \mathcal{C} \cup \partial\triangle$. We assume that the gradient field $\nabla f(1,.,.)$ of the inhomogeneous polynomial does not vanish in $\triangle$. Furthermore a curve $\mathcal{V}(\mathbf{p}) \subset \triangle$ is given by its parametric representation $\mathbf{p} = \mathbf{p}(u)$, $u \in [0,1]$. If*

[4] The symmetric version is $\max(\mathrm{HD}_{\triangle}(\mathcal{V}(\mathbf{p}), \mathcal{C}^*), \mathrm{HD}_{\triangle}(\mathcal{C}^*, \mathcal{V}(\mathbf{p})))$.

$$c \leq ||\ \nabla f(1,.,.)|_{(x_1,x_2)}\,||_{\mathbb{R}^2}\ \forall\ (x_1,x_2) \in \triangle \quad \textit{and} \quad ||\mathrm{eval}(f,\mathbf{p})|| \leq \epsilon$$

and

$$p_0(u) \geq N \quad \forall u \in [0,1]$$

hold, where c, ϵ and N are certain positive constants, then the one–sided Hausdorff distance can be bounded by

$$\mathrm{HD}_{\triangle}(\mathcal{V}(\mathbf{p}),\mathcal{C}^*) \leq \frac{\epsilon}{c\,N^d}. \tag{10}$$

Proof. The proof is a consequence of the mean value theorem, which is applied to the integral curves of the gradient field emanating from the parametric curve and hitting the implicit curve or the boundary of the triangle. More precisely, we consider the integral curve $\gamma(t) := (x_1(t), x_2(t))$ of the normalized gradient field $V = \nabla\bar{f}/||\nabla\bar{f}||$ of $\bar{f}(x_1,x_2) = f(1,x_1,x_2)$. We choose the starting point $\gamma(0)$ of the integral curve $\gamma(t)$ to lie on the parametric curve $\mathbf{p}$. In the absence of points with a vanishing gradient in the domain of interest, $\gamma(t)$ hits the implicit curve or the boundary of the triangle in $\gamma(s)$ for some $s \in \mathbb{R}$. Since the curve is parameterized by its arc length,

$$\|\gamma(s) - \gamma(0)\| \leq s.$$

Let $F(t)$ be the restriction of $\bar{f}$ to $\gamma(t)$:

$$F(t) := \bar{f}(x_1(t), x_2(t)).$$

Applying the mean value theorem we obtain

$$\exists \xi \in]0,s[: \quad \frac{F(s)-F(0)}{s-0} = F'(\xi)$$

Due to $F'(t) = \dot{\gamma}(t)\cdot\nabla\bar{f}\big|_{\gamma(t)} = \|\nabla\bar{f}\big|_{\gamma(t)}\|$ we obtain

$$\frac{|F(s)-F(0)|}{|s-0|} = |F'(\xi)| = \|\nabla\bar{f}\big|_{\gamma(\xi)}\|, \ \text{ hence } \ s = \frac{|F(s)|}{\|\nabla\bar{f}(\gamma(\xi))\|} = \frac{|\bar{f}(\gamma(s))|}{\|\nabla\bar{f}(\gamma(\xi))\|}$$

Finally, we observe that the values

$$f\left(1, \frac{p_1(u)}{p_0(u)}, \frac{p_2(u)}{p_0(u)}\right), \quad u \in [0,1], \tag{11}$$

of the values of the inhomogeneous implicit representation along the parametric curve are bounded by ϵ/N^d, which completes the proof. □

The next step is the computation of the constants c and N which are needed in Lemma 1. One can see that for curves that have singular points in the domain of interest, (10) is not defined, since the gradient vanishes. Consequently, such cases have to be excluded. More precisely, vanishing gradients correspond to

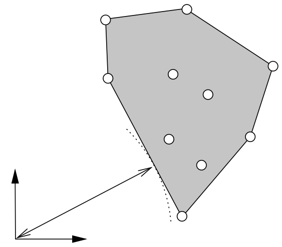


Fig. 1. Bounding the minimal norm of the gradient.

singular points (including isolated points) of the original curve or of other iso-value curves ("algebraic offsets").

In the regular case, a lower bound for the minimal gradient can be computed. The essential ingredient of this algorithm is the convex hull property of Bernstein–Bézier representations. Clearly, this algorithm gives only a conservative lower bound on $||\nabla f(1,.,.)||$. The result can be made more accurate by splitting the domain into smaller triangles.

Algorithm 2
Input: control points c_{ijk} of a bivariate polynomial.
Output: lower bound for the minimal gradient

1. Compute the partial derivatives of $f(1, x_1, x_2)$ with respect to x_1 and x_2.
2. Describe them in Bernstein–Bézier form with respect to the domain triangle.
3. Combine the corresponding coefficients of the derivative patches together to vector-valued control points $d_{ijk} \in \mathbb{R}^2$
4. Compute the minimal distance from the origin to the convex hull of the d_{ijk}, see Figure 1.
5. This distance serves as constant c in Lemma 1.

Remark 3. The procedure can be generalized to the surface case. While the algorithms for the three-dimensional convex hull computations are more involved and need special data structures for the storage of the data points [GO04], the time complexity is still the same as in the planar case.

Remark 4. The constant N can be chosen as the minimum value of the 0–th components of the control points in (1), provided that all of them are positive. More precisely, the so–called 'weights' of the rational curve have to be positive, and N can be chosen as the minimum weight.

5 Distance bound for approximate implicitization

Based on the previous result, we derive an upper bound on the one–sided Hausdorff distance between the parametric curve and any approximate implicitization which has a certain accuracy.

5.1 Estimating the implicitization error

In order to estimate the Haussdorff distance of a parametric and an approximate implicit curve we use a known result [SS04] that allows to estimate the error in the coefficient vector in terms of the condition number K.

Lemma 2. *Let* $\mathbf{p}$ *be a triple of polynomials of parametric degree* n*, with* $\|\mathbf{p}\| = 1$*. Furthermore, let* $f_1, f_2 \in I$ *be polynomials of degree* d *with* $\|f_1\| = \|f_2\| = 1$ *such that* $\|\mathrm{eval}(f_1, \mathbf{p})\| \leq \epsilon$ *and* $\|\mathrm{eval}(f_2, \mathbf{p})\| \leq \epsilon$*. Then we have one of the following:*

$$\|f_1 - f_2\| \leq K \cdot 4 \cdot \epsilon, \quad \|f_1 + f_2\| \leq K \cdot 4 \cdot \epsilon,$$

where K *is the condition number of* $\mathbf{p}$*.*

Proof. The complete proof (of a more general result) is given in [SS04]. Here we restrict ourselves to a sketch of the proof. Let f_3 be such that $\|f_3\| = 1$ and $\|\mathrm{eval}(f_3, \mathbf{p})\|$ is minimal. It follows that $\|\mathrm{eval}(f_3, \mathbf{p})\| \leq \|\mathrm{eval}(f_1, \mathbf{p})\| \leq \epsilon$. In first order approximation we have, that

$$r_1 := f_1 - f_3, \quad \text{and} \quad r_2 := f_3 - f_2,$$

are in $f_3^{\perp}$. In order to estimate $\|r_i\|$, $i = 1, 2$, we get

$$\|r_i\| = K \cdot \|\mathrm{eval}(r_i, \mathbf{p})\| \leq K \cdot \epsilon.$$

It follows, that

$$\|f_1 - f_2\| \leq \|f_1 - f_3\| + \|f_3 - f_2\| \leq K \cdot 2 \cdot \epsilon.$$

If we take terms of higher order into account, then we get the inequality $\|f_1 - f_2\| \leq 4 \cdot K \cdot \epsilon$, see [SS04]. □

5.2 Bounding the minimal gradient

From the previous Section we can conclude that the output of the implicitization process is no longer an exact polynomial, but that its coefficients can only be specified up to a certain tolerance. This means that the possible outputs of the implicitization process form a whole set of curves.

We denote the set of defining polynomials of the implicitized curves by

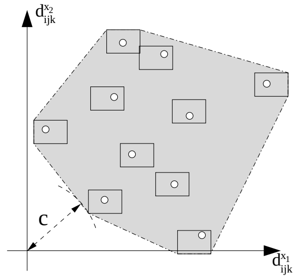


Fig. 2. Bounding the minimal norm of the gradient of a perturbed polynomial. The rectangles represent the areas where all possible control points lie in.

$$F_\epsilon := \{f \mid \|\text{eval}(f, \mathbf{p})\| \le \epsilon, \|f\| = 1\}.$$

The corresponding coefficients of all $f \in F_\epsilon$ lie in certain intervals. The length of these intervals can be bounded using Lemma 2.

On the other hand, Lemma 1 allows us to bound the Hausdorff distance between a parametric and an algebraic curve. In order to get a distance bound between a parametric and an arbitrary $f \in F_\epsilon$ we need to compute a lower bound for the minimal gradient for all possible $f \in F_\epsilon$. This bound is given by

$$G := \min_{f \in F_\epsilon} \min_{(x_1,x_2) \in \Omega} \| \nabla f(1, ., .)|_{(x_1,x_2)} \|.$$

In order to compute this bound the same technique as in Section 4 can be applied. One has to replace the exact control points by intervals. Hence, standard techniques for interval arithmetics have to be applied for computing the d_{ijk}. These are no longer points in $\mathbb{R}^2$, but rectangles containing all possible positions of the control points. Consequently, one has to compute the convex hull of these rectangles in order to gain a lower bound for the minimal gradient, cf. Figure 2. For further informations on interval arithmetics and related techniques see [SLMW03] and the references cited therein.

Algorithm 3 ("Minimal Gradient")
Input: parametric representation $\mathbf{p}$ of a curve, $\epsilon > 0$
Output: G

1. Compute K using Algorithm "Condition Number".
2. Compute the bound given in Lemma 2 and an approximate implicitization f for $\mathbf{p}$.
3. Generate intervals using the coefficients of f as center and adding/ subtracting the bound derived in the previous step.
4. Determine the derivative patches of $f(1, x_1, x_2)$ in x_1 and x_2 direction and describe them in Bernstein–Bézier (with interval coefficients !) form with respect to the domain triangle.

curve	K	G	position err.
cardioid	$0.43523 \cdot 10^6$	0	∞
bicorn	$0.19899 \cdot 10^9$	0	∞
trifolium	2478.62	0	∞
tacnode	12.687	0.17537	$0.98718 \cdot 10^{-5}$

Table 3. Geometric robustness and position bound

5. For all pairs of corresponding coefficients of the derivative patches, generate the Cartesian product of the intervals.
6. Collect the vertices of all these rectangles and determine their convex hull.
7. The shortest distance from the origin to this convex hull serves as G. (If the convex hull contains the origin, then $G = 0$)

5.3 Perturbation regions of parametric curves

In this Section we combine the results of the previous parts and determine an upper bound for the Hausdorff distance between an exact parametric and approximatively computed implicit curve.

Theorem 1. *Let* $\mathbf{p}$ *with* $\|\mathbf{p}\|$ *be a triple of polynomials of degree less or equal than* n*, and let* $\mathcal{V}(\mathbf{p}) \subset \triangle$ *be the curve defined by* $\mathbf{p}$*. If the bound* G *computed by Algorithm 3 is nonzero, then for any algebraic curve segment* $\mathcal{C} \subset \triangle$ *defined by an* $f \in F_\epsilon$ *of degree* d*,*

$$\mathrm{HD}_\triangle(\mathcal{V}(\mathbf{p}), \mathcal{C}^*) \leq \frac{\epsilon}{GN^d},$$

where the constant N *is defined as in Lemma 1.*

Proof. The proof is an immediate consequence of the previous two lemmas. □

The bound given in Theorem 1 defines two offset curves to $\mathbf{p}$ that enclose a perturbation region within the triangle. For any approximate parameterization $\mathbf{p}$ of a curve and for any approximate implicitization with a given precision ϵ the obtained curve $\mathcal{C}$ lies within this perturbation region.

Clearly, the result of this Theorem is only meaningful for points that are further away from the boundary than $\frac{\epsilon}{GN^d}$. This is due to the fact, that we do not only measure the distance to the implicitly defined curve, but also to the boundary of the triangle, see Figure 3.

Example 2. In Table 3 we determine for each of the four examples the bound provided in Theorem 1. Using Algorithm 3 we compute a lower bound G for the minimal gradient. In the examples we set $\epsilon = 10^{-6}$.

In the first two cases the high condition number K reflects the fact that the implicitization process is very unstable. Consequently, the coefficient bounds

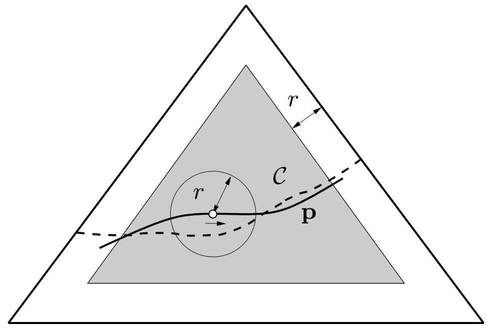


Fig. 3. For each point that lies on **p** and in the grey shaded triangle exists within a certain bound $r := \epsilon/(GN^d)$ a point on the implicit curve $\mathcal{C}$.

are very poor and the bound for the minimal gradient yields zero. No prediction for the position error of the obtained implicit curve can be made.

For the trifolium the implicitization is robust but the obtained implicit representation is unstable with respect to small errors in the coefficients. Again the geometric robustness is poor and the position bound is infinity.

In the last example the implicitization as well as the obtained implicit representation are stable under numerical perturbations; the geometrical robustness is good. Knowing the precision of the implicitization process we are able to predict the maximal displacement of the implicit curve.

6 Conclusion

Schicho and Szilágyi [SS04] have analyzed the robustness of approximate implicitizations with respect to the resulting coefficients. On the other hand, Aigner et al. [AJK04] treated the geometric robustness of implicit representations if some error in the coefficient vector is allowed. In the present paper, we combined these results, in order to predict the geometric stability of approximate implicitization. More precisely, given a parametric representation, we may derive an error estimate for the obtained coefficients of the implicit curve. With this result we can compute a bound such that any approximate implicitization lies within a certain neighborhood of the original parametric curve. The width of this vicinity is determined by the bound. While we have presented the results for curves, they can be generalized immediately to the case of general hypersurfaces.

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