On the existence of biharmonic tensor–product Bézier surface patches

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Abstract

A tensor–product Bézier surface patch x of degree (m, n) is called biharmonic if it satisfies $\Delta^2 x = 0$. As shown by Monterde and Ugail (2004), these surface patches are fully determined by their four boundaries. In this note we derive necessary conditions for their existence.

Key words: Bilaplacian operator, biharmonic surfaces

Monterde and Ugail (2004) discuss in Chapter 3 the construction of a tensor–product surface patch of degree (m, n), with $m, n \geq 4$,

$$x(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{c_{i,j}}{i!j!} u^i v^j, \quad (u, v) \in [0, 1]^2, \tag{1}$$

with unknown coefficients $c_{i,j}$, which satisfies the biharmonic condition

$$\Delta^2 x = \frac{\partial^4 x}{\partial u^4} + 2 \frac{\partial^4 x}{\partial u^2 \partial v^2} + \frac{\partial^4 x}{\partial v^4} = 0 \tag{2}$$

and which matches the four given boundary curves

$$\begin{aligned} x(0, v) &= \sum_{j=0}^{n} p_j v^j, & x(1, v) &= \sum_{j=0}^{n} q_j v^j, \\ x(u, 0) &= \sum_{i=0}^{m} r_i u^i, \text{ and } & x(u, 1) &= \sum_{i=0}^{m} s_i u^i \end{aligned} \tag{3}$$

with given coefficients p_j, q_j, r_i, s_i. Without loss of generality we shall assume that the degrees satisfy $n \leq m$.

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The given coefficients of the boundary curves have to satisfy

$$\begin{aligned} x(0,0) &= p_0 = r_0, & x(1,0) &= q_0 = \sum_{i=0}^{m} r_i, \\ x(0,1) &= \sum_{j=0}^{n} p_j = s_0, \quad \text{and} & x(1,1) &= \sum_{j=0}^{n} q_j = \sum_{i=0}^{m} s_i. \end{aligned} \tag{4}$$

By comparing the coefficients, Monterde and Ugail (2004) translate the biharmonic condition into a system of linear equations of the form

$$c_{k+4,l} + 2\,c_{k+2,l+2} + c_{k,l+4} = 0, \quad k = 0,\ldots,m, \quad l = 0,\ldots,n \tag{5}$$

with $c_{k,l} = 0$ if $k > m$ or $l > n$.

The boundary curves $x(0,v)$ and $x(u,0)$ determine the coefficients

$$p_j = c_{0,j}, \quad j = 0,\ldots,n, \quad \text{and} \quad r_i = c_{i,0}, \quad i = 0,\ldots,m. \tag{6}$$

The remaining two boundary curves $x(1,v)$ and $x(u,1)$ determine the row and column sums of the coefficients $c_{i,j}$,

$$q_j = \sum_{i=0}^{m} \frac{c_{i,j}}{i!j!}, \quad j = 0,\ldots,n, \quad \text{and} \quad s_i = \sum_{j=0}^{n} \frac{c_{i,j}}{i!j!}, \quad i = 0,\ldots,m. \tag{7}$$

As shown by Monterde and Ugail (2004), the equations (5) – (7) suffice to determine all unknown coefficients $c_{i,j}$.

In this note we shall study the existence of solutions, which has not been analyzed so far. It will be shown that the biharmonicity conditions (5) imply certain constraints for the choice of the boundary curves.

For any two arbitrary but fixed indices $i_0 \in \mathbb{Z}_+$ and $j_0 \in \{0,1\}$, we consider the sequence

$$c_{i_0-2\,k,j_0+2\,k}, \quad \text{where} \quad k = 0,\ldots,\lfloor \frac{i_0}{2} \rfloor. \tag{8}$$

If any two coefficients of such a sequence possess indices not in $\{0,\ldots,m\} \times \{0,\ldots,n\}$, then all coefficients of the sequence vanish. Indeed, the biharmonicity conditions form a tridiagonal system of equations for the coefficients of the sequence, and the two coefficients, whose indices are not in $\{0,\ldots,m\} \times \{0,\ldots,n\}$, add two equations which force the solution to be the trivial one. The consequences are visualized in Figures 1 and 2.

In order to analyze the resulting conditions for the boundary curves we need to distinguish between several cases.

Case 1: n is even. This case is illustrated by Fig. 1, where $n = 4$.

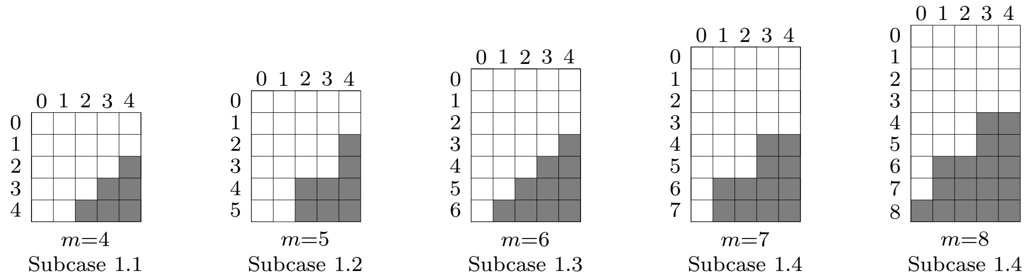

Fig. 1. Matrices of coefficients $c_{i,j}$ (schematic) for $n = 4$. The grey boxes correspond to coefficients which have to vanish, due to (5).

Subcase 1.1: $m = n$**.** All coefficients $c_{i,j}$ with $i + j \geq n + 2$ vanish. This does not imply any conditions for the given boundary curves.

Subcase 1.2: $m = n + 1$**.** All coefficients with $2\left\lfloor \frac{i}{2} \right\rfloor + j \geq n + 2$ vanish. This does not imply any conditions for the given boundary curves.

Subcase 1.3: $m = n + 2$**.** All coefficients $c_{i,j}$ with $i + j \geq n + 3$ vanish. Due to $c_{m,j} = 0$ for $j = 1, \ldots, n$, the equations (6) and (7) for r_m and s_m lead to an additional condition for the given boundary curves. A biharmonic patch exists only if the given boundaries satisfy

$$r_{n+2} = s_{n+2} \;\text{ or, equivalently, }\; \left.\frac{\partial^{n+2}}{\partial u^{n+2}} x(u, v)\right|_{(0,0)} = \left.\frac{\partial^{n+2}}{\partial u^{n+2}} x(u, v)\right|_{(0,1)}. \quad (9)$$

Subcase 1.4: $m \geq n + 3$**.** All coefficients $c_{i,j}$ with $2\left\lfloor \frac{i}{2} \right\rfloor + j \geq n + 3$ vanish. Due to $c_{n+2,j} = c_{n+3,j} = 0$ for $j = 1, \ldots, n$, the equations (6) and (7) for r_{n+2}, r_{n+3}, s_{n+2} and s_{n+3} lead to two additional conditions for the given boundary curves. A biharmonic patch exists only if the given boundaries satisfy (9) and

$$r_{n+3} = s_{n+3} \;\text{ or, equivalently, }\; \left.\frac{\partial^{n+3}}{\partial u^{n+3}} x(u, v)\right|_{(0,0)} = \left.\frac{\partial^{n+3}}{\partial u^{n+3}} x(u, v)\right|_{(0,1)}. \quad (10)$$

Moreover, due to $c_{i,j} = 0$ for $i = n + 4, \ldots, m$ and $j = 0, \ldots, n$, a biharmonic patch exists only if the given boundaries satisfy

$$r_i = s_i = 0 \;\text{ or, equivalently, }\; \left.\frac{\partial^{i}}{\partial u^{i}} x(u, v)\right|_{(0,0)} = \left.\frac{\partial^{i}}{\partial u^{i}} x(u, v)\right|_{(0,1)} = 0, \quad (11)$$

for $i = n + 4, \ldots, m$. Clearly, this condition is active only if $m > n + 3$.

Case 2: n is odd. This case is illustrated by Fig. 2, where $n = 5$.

Subcase 2.1: $m = n$**.** All coefficients $c_{i,j}$ with $2\left\lfloor \frac{i}{2} \right\rfloor + j \geq n + 1$ vanish. This does not imply any conditions for the given boundary curves.

Subcase 2.2: $m = n + 1$**.** All coefficients $c_{i,j}$ with $2\left\lceil \frac{i}{2} \right\rceil + j \geq n + 3$ vanish. This does not imply any conditions for the given boundary curves.

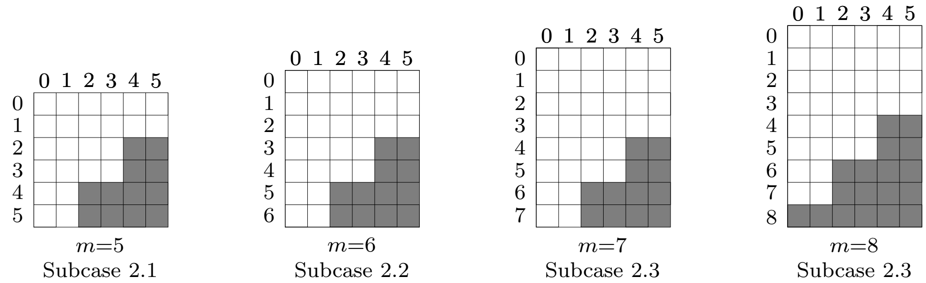

Fig. 2. Matrices of coefficients $c_{i,j}$ (schematic) for $n = 5$. The grey boxes correspond to coefficients which have to vanish, due to (5).

Subcase 2.3: $m \geq n+2$**.** All coefficients $c_{i,j}$ with $2\lfloor \frac{i}{2} \rfloor + j \geq n+3$ vanish. Due to $c_{i,j} = 0$ for $i = n+3, \ldots, m$ and $j = 0, \ldots, n$, a biharmonic patch exists only if the given boundaries satisfy (11) for $i = n+3, \ldots, m$. Clearly, this condition is active only if $m > n+2$.

Table 1 summarizes the necessary conditions for the existence of biharmonic tensor–product patches of degree (m, n). If the difference between the degrees m and n is too large, then no biharmonic patch exists, unless the boundaries satisfy certain compatibility conditions, as described in the previous analysis. Clearly, a solution of the higher degree $(\max(m,n), \max(m,n))$ can always be found.

Table 1
Existence conditions for biharmonic patches.

- $\emptyset$: no condition,
- A: condition (9),
- B: conditions (9) and (10),
- $\leftarrow, \uparrow$: a solution exists only if the conditions of the case pointed at by the arrow are satisfied. Each arrow adds two conditions (11).

m \ n	4	5	6	7	8	9	10	11	
4	$\emptyset$	$\emptyset$	A	B	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	
6	A	$\emptyset$	$\emptyset$	$\emptyset$	A	B	$\leftarrow$	$\leftarrow$	
7	B	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\leftarrow$	$\leftarrow$	$\vdots$
8	$\uparrow$	$\uparrow$	A	$\emptyset$	$\emptyset$	$\emptyset$	A	B	
9	$\uparrow$	$\uparrow$	B	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
10	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	A	$\emptyset$	$\emptyset$	$\emptyset$	
11	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	B	$\emptyset$	$\emptyset$	$\emptyset$	
					$\ldots$				

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