A Construction of Rational Manifold Surfaces of Arbitrary Topology and Smoothness from Triangular Meshes

Giovanni Della Vecchia, Bert Jüttler^a and Myung-Soo Kim^b

^aJohannes Kepler University Linz, Institute of Applied Geometry, Austria ^bSeoul National University, School of Computer Science and Engineering, South Korea

Abstract

Given a closed triangular mesh, we construct a smooth free–form surface which is described as a collection of rational tensor–product and triangular surface patches. The surface is obtained by a special manifold surface construction, which proceeds by blending together geometry functions for each vertex. The transition functions between the charts, which are associated with the vertices of the mesh, are obtained via subchart parameterization.

Keywords. Manifold surface, geometric continuity, smooth free-form rational surface, arbitrary topological genus.

1 Introduction

Methods for representing closed surfaces of arbitrary topology by surfaces with explicitly available parametric representations (i.e., no subdivision surfaces) are a classical topic in Computer Aided Geometric Design. The existing techniques can roughly be organized in two groups: patch-based methods and manifold-type constructions.

The patch–based methods exploit the concept of geometric continuity in order to build smooth surfaces by joining polynomial or rational surface patches with various degrees of geometric continuity. A survey of this concept – with

Email addresses: {giovanni.dellavecchia|bert.juettler}@jku.at (Giovanni Della Vecchia, Bert Jüttler), mskim@cse.snu.ac.kr (Myung-Soo Kim).

a particular focus on constructive realizations – has been given by Peters (2002a). Here we list only a few representative references.

Reif (1998) introduced topologically unrestricted rational B-splines (TURBS) which use singularly parameterized surfaces in order to deal with situations where three or more than four quadrilateral surface patches meet in a common point. Prautzsch (1997) avoids the use of singular points by composing the parameterization of the geometry at extraordinary points with piecewise polynomial reparameterizations of the parameter domain. Peters (2002b) describes a construction of curvature continuous free-form surfaces of degree 2 which uses tensor-product patches of degree (3, 5), which can be generalized to a G^s construction of degree (s+1, d+2s-2), where d controls the flexibility at extraordinary points.

The patch–based constructions are able to generate smooth free-form surfaces of relatively low degree. Typically they require a special treatment for "extraordinary" points, i.e., points where other than four quadrangular or other than six triangular patches meet.

The manifold-type constructions are based on a different paradigm, which is taken from differential geometry. The surface is covered with overlapping charts, and transition functions are defined between them within the overlapping regions. The transition functions have to possess the same order of smoothness as the final surface. Also, the transition functions have to satisfy the cocycle condition in regions where more than two charts overlap.

As a conceptual advantage, the blending approach provides a natural way for splitting the modeling problem into smaller and simpler subproblems. The manifold framework makes it particularly simple to define auxiliary linear spaces of scalar and vector–valued fields on the surface, since the construction works independently of the dimension of the embedding space. This is useful for surface fitting and for applications involving partial differential equations on surfaces.

If the charts and the transition functions are known, then two different techniques can be used to define the manifold. The first one relies on the control point paradigm, by defining locally supported blending functions on the manifold. The second is a blending approach, which proceeds by defining geometry functions for each chart and blending them together via influence functions.

Grimm and Hughes (1995) were the first who presented a constructive manifold surface construction. The desired surface is specified using a sketch mesh where all vertices have valence four. Charts are created for each element of the mesh (vertices, edges, faces) and the transition functions are created by blending projective mappings. The construction is based on the control point paradigm. An alternative manifold construction has been presented by Cotrina-Navau and Pla-Garcia (2000). They first use subdivision to isolate the extraordinary vertices. The charts are created via the characteristic map of the vertex. Cotrina-Navau et al. (2002) present a theoretical approach and describe several realizations of a generic scheme.

Ying and Zorin (2004) presented a novel construction for creating manifold surfaces from quadrangular meshes. The charts, which are associated with the vertices of the mesh, are special *n*-gons with curved boundaries. The transition functions are chosen from a particular class of holomorphic functions (involving complex-valued roots) which have the property to contain both a function and its inverse function. The manifold surface is then obtained via blending, yielding a C^{∞} smooth surface with explicit non-singular parameterizations. Combined with displacement mappings, Yoon (2006) has used manifold splines of this type for modeling complex free-form objects.

In an unpublished technical report, Wagner et al. (2003) describe a C^2 construction with piecewise polynomial patches.

The manifold splines of Gu et al. (2005, 2007) are based on an affine atlas which is computed from a given triangular mesh. It requires the introduction of holes in this mesh (and the associated surface), in order to guarantee the existence of the atlas. The holes are then dealt with by traditional hole–filling techniques.

Another variant of manifold surfaces uses a simple base manifold to parameterize all closed smooth surfaces of a given genus. Any smooth surface with that genus can then be obtained as an embedding of this base manifold. Wallner and Pottmann (1997) construct such a base manifold by considering equivalence classes of points in the hyperbolic, elliptic or Euclidean plane with respect to certain discrete subgroups of the corresponding motion group. Grimm (2002, 2004) describes a construction which is based on embedded manifolds. Clearly, if this approach is adopted, then modifications of the topological genus ("adding handles") imply changes of the parameterization manifold.

This paper presents a construction of rational blending manifolds from a given triangular mesh. The charts are chosen as circular disks, and they are associated with the vertices of the mesh. The edges and faces of the mesh correspond to overlapping regions of two and three charts, respectively. The general layout is shown in Fig. 1.

We define the transition functions between the charts by parameterizing the subcharts over common parameter domains. In this way we are not restricted to special classes of transition functions, such as special holomorphic functions as used by Ying and Zorin (2004). The manifold surface is obtained following the blending approach, where the influence functions can be obtained by taking



Fig. 1. A triangular mesh and the associated chart layout. The blue and red regions correspond to overlapping regions of two and three charts, respectively.



Fig. 2. A triangular mesh with 136 vertices and 288 faces describing a hollow cube (left) and the associated C^2 smooth blending manifold surface (right). The surface was rendered using 18,144 triangles.

suitable powers of the equation of the unit circle. As a first example, Fig. 2 shows a C^2 manifold surface describing a hollow cube.

The remainder of this paper is organized as follows. Section 2 introduces the notion of blending manifolds which are associated with triangular meshes. Section 3 presents a particular construction for rational blending manifolds. Section 4 presents several examples obtained from the presented construction that illustrate the influence of the shape parameters and demonstrate the smoothness of the surfaces. Finally we conclude this paper.

2 Blending manifolds associated with triangular meshes

Given a triangular mesh, we define the notion of an associated parameterized atlas. We then use this manifold structure and additional geometry functions and influence functions for each chart to define a blending manifold surface. This surface can achieve an arbitrary order of smoothness.

2.1 Charts and subcharts

We consider a given triangular mesh M in \mathbb{R}^3 , where m_V is the number of vertices, m_F is the number of faces and m_E is the number of edges. Let

$$V = \{i : i = 1, \dots, m_V\}$$
(1)

be the **set of vertex indices**. The mesh is assumed to describe the boundary of a compact set. The faces and vertices of the mesh are oriented by outward-pointing normals. We use the mesh M to define the charts and the transition functions of the manifold. More precisely, we define a *chart* for each *vertex* of the mesh.

For the *i*-th vertex of the mesh, we denote the surrounding vertices in counterclockwise order by $n(i, 1), \ldots, n(i, v(i))$, where v(i) is the valence of the vertex. The second index of *n* is counted modulo v(i), i.e. n(i, j) = n(i, j + v(i)). For each vertex, let

$$N(i) = [n(i,1), \dots, n(i,v(i))], \quad i \in V$$
(2)

be the ordered list of neighboring vertices. In addition to the set V of vertex indices, we define the **set of edge indices**

$$E = \{\{i, n(i, r)\}: i \in V, r = 1, \dots, v(i)\}.$$
(3)

and the set of face indices

$$F = \{\{i, n(i, r), n(i, r+1)\}: i = 1, \dots, m_V, r = 1, \dots, v(i)\}.$$
(4)

Remark 1 Any edge index $e \in E$ is a set containing two vertex indices, i.e., $e = \{i, j\} = \{j, i\}$. Similarly, any face index $f \in F$ is a set containing three vertex indices, $f = \{i, j, k\} = \{j, k, i\} = \ldots = \{k, j, i\}$.



Fig. 3. A chart C^i with edge subcharts C^i_j, C^i_k, C^i_l , face subcharts C^i_{jk}, C^i_{kl} , and innermost part \hat{C}^i . The thick curve is the union of the three boundary arcs $(\hat{C}^i \cap C^i_k) \cup (C^i_j \cap C^i_{jk}) \cup (C^i_l \cap C^i_{kl})$; it is assumed to be a C^s smooth curve.

We define a system of charts C^i and subcharts $C^i_j, C^i_{jk} \subset C^i$ which can be associated with the given triangular mesh. All charts and subcharts are closed subsets of \mathbb{R}^2 . The generic layout of subcharts is shown in Fig. 3.

Remark 2 Throughout this paper, C^s refers to the order s of smoothness. We say that a function is C^s smooth if it is s-times continuously differentiable. The symbols C^i , C^i_j and C^i_{jk} are used to denote the charts and subcharts of the atlas. As usual, \overline{D} indicates the closure of a set $D \subset \mathbb{R}^2$.

Definition 3 A set $\{C^i : i \in V\}$ of compact, simply connected subsets C^i of \mathbb{R}^2 with \mathcal{C}^s smooth boundaries will be called a **system of charts** associated with the triangular mesh M. For each chart C^i we define v(i) edge subcharts $C^i_{n(i,r)}$ and v(i) face subcharts $C^i_{n(i,r),n(i,r+1)}$, where $r = 1, \ldots, v(i)$. All edge and face subcharts are quadrangles and triangles with curved edges, respectively. For any pair of subcharts, the intersection of the interior parts is empty. Let

$$\hat{C}^{i} = \overline{C^{i} \setminus (\bigcup_{r=1,...,v(i)} C^{i}_{n(i,r)} \cup \bigcup_{r=1,...,v(i)} C^{i}_{n(i,r),n(i,r+1)})}$$
(5)

be the remaining or **innermost part** of C^i .

For any r, let k = n(i,r), l = n(i,r-1) and j = n(i,r+1). The edge subcharts and face subcharts satisfy the following conditions.

- (i) The subchart C_{jk}^i shares one boundary arc with C_j^i and one with C_k^i . The remaining boundary arc is contained in ∂C^i .
- (ii) The subchart C_k^i shares one boundary arc with C_{jk}^i , one with C_{kl}^i and one with \hat{C}^i . The remaining boundary arc is contained in ∂C^i .
- (iii) The union of the three boundary arcs $(\hat{C}^i \cap C_k^i) \cup (C_j^i \cap C_{jk}^i) \cup (C_l^i \cap C_{kl}^i)$ is a \mathcal{C}^s smooth curve.
- (iv) The sequence of edge and face subcharts



Fig. 4. The boundary conditions for edge subchart parameterizations.

$$C_{n(i,1)}^{i}, C_{n(i,1),n(i,2)}^{i}, C_{n(i,2)}^{i}, C_{n(i,2),n(i,3)}^{i}, \dots, C_{n(i,v(i))}^{i}, C_{n(i,v(i)),n(i,1)}^{i}$$
(6)

is arranged in counterclockwise order along ∂C^i .

The face subcharts are triangular regions that correspond to the overlap of three charts. The edge subcharts, along with the two neighboring face subcharts, define biangular regions that correspond to the overlap of two charts.

2.2 Parameterization of subcharts

We introduce edge subchart parameterizations and face subchart parameterizations, which will then be combined in order to define the transition functions. First we define *edge subchart parameterizations*. Their domain is the unit square, which will be denoted by $\Box = [0, 1]^2$. Let

$$E_1 = \{0\} \times [0,1], E_2 = [0,1] \times \{0\}, E_3 = \{1\} \times [0,1], E_4 = [0,1] \times \{1\}$$
 (7)

be the left, lower, right and upper edge of \Box , respectively.

Definition 4 For each edge $e = \{i, j\} \in E$ of the given triangular mesh we consider two mappings $\phi_j^i : \Box \to C_j^i$ and $\phi_j^j : \Box \to C_i^j$. These mappings are called **edge subchart parameterization** provided that they are \mathcal{C}^s smooth, surjective, orientation preserving (hence also regular) and satisfy

$$\phi_j^i(E_1) = C_j^i \cap \hat{C}^i , \qquad \phi_j^i(E_3) = C_j^i \cap \partial C^i, \\
\phi_i^j(E_1) = C_i^j \cap \partial C^j \qquad and \qquad \phi_i^j(E_3) = C_i^j \cap \hat{C}^j.$$
(8)

Remark 5 The two conditions (per edge subchart parameterization) imply that E_2 and E_4 are mapped to the *lower* and *upper* boundaries of C_j^i and C_i^j , as the mappings preserve the orientation.

See Fig. 4 for an illustration of this definition.



Fig. 5. Boundary conditions for face subchart parameterizations.

Similarly we define *face subchart parameterizations*. Their domain is the standard triangle

$$\Delta = \{ (x, y, z) : \quad x \ge 0, \ y \ge 0, \ z \ge 0, \ x + y + z = 1 \} \subset \mathbb{R}^3.$$
(9)

Let T_1 , T_2 and T_3 be the edges of the triangle which are contained in the zx, xy and yz plane, respectively. The points in the standard triangle correspond to the barycentric parameters which are used for triangular Bézier patches.

Definition 6 For each face $f = \{i, j, k\} \in F$ of the triangular mesh we consider three mappings

$$\phi^i_{jk} : \triangle \to C^i_{jk}, \ \phi^j_{ki} : \triangle \to C^j_{ki}, \ and \ \phi^k_{ij} : \triangle \to C^k_{ij}.$$
 (10)

These mappings are called **face subchart parameterizations** provided that they are C^s smooth, surjective, orientation preserving (hence also regular) and satisfy

$$\phi_{jk}^{i}(T_{3}) = C_{jk}^{i} \cap \partial C^{i}, \ \phi_{ki}^{j}(T_{1}) = C_{ki}^{j} \cap \partial C^{j}, \ and \ \phi_{ij}^{k}(T_{2}) = C_{ij}^{k} \cap \partial C^{k}.$$
 (11)

Remark 7 Similar to Remark 5, the three conditions imply that the remaining edges are mapped as shown in Fig. 5.

This definition is illustrated by Fig. 5. The lower index of a face subchart parameterization is considered to be a set-valued one (similar to the case of edge indices), i.e.,

$$\phi_{jk}^{i} = \phi_{\{j,k\}}^{i} = \phi_{\{k,j\}}^{i} = \phi_{kj}^{i}.$$
(12)

In order to keep the notation simple, we omit the brackets in the index.



Fig. 6. Left: Vertex *i* with neighbors. Right: The overlapping regions between the charts C^i and C^j and the transition functions Φ^{ij} and Φ^{ji} .

2.3 Transition functions and atlas

For any edge $e = \{i, j\}$ we consider the two neighboring triangular faces $\{i, j, k\}$ and $\{i, j, l\}$ in F, where the vertex i has the ordered neighbors $N(i) = [\ldots, k, j, l, \ldots]$, see Fig. 6, left. We consider the subcharts C_j^i , C_{lj}^i and C_{jk}^i of C^i and the subcharts C_i^j , C_{li}^j and C_{jk}^j of C^j as shown in Fig. 6, right.

Definition 8 The two sets

$$O_j^i = C_{lj}^i \cup C_j^i \cup C_{jk}^i \subset C^i \quad and \quad O_i^j = C_{li}^j \cup C_i^j \cup C_{ik}^j \subset C^j$$
(13)

are called the **overlapping regions** between the charts C^i and C^j in the chart C^i and in the chart C^j , respectively. The **transition function** between C^i and C^j is defined by

$$\Phi^{ij}: \ O_{j}^{i} \to O_{i}^{j}: \ x \mapsto \Phi^{ij}(x) = \begin{cases} (\phi_{li}^{j} \circ (\phi_{lj}^{i})^{-1})(x) & \text{if} \quad x \in C_{lj}^{i} \\ (\phi_{i}^{j} \circ (\phi_{j}^{i})^{-1})(x) & \text{if} \quad x \in C_{j}^{i} \\ (\phi_{ik}^{j} \circ (\phi_{jk}^{i})^{-1})(x) & \text{if} \quad x \in C_{jk}^{i} \end{cases}$$
(14)

The subchart parameterizations ϕ_j^i and ϕ_{jk}^i are said to be **valid** if all transition functions Φ^{ij} are C^s smooth.

Remark 9 By definition, the transition functions are bijective and continuous. In addition, they satisfy the cocycle condition, i.e., $\Phi^{jk} \circ \Phi^{ij} = \Phi^{ik}$. In addition we need to ensure \mathcal{C}^s -smoothness, in order to make them valid. In the second part of the paper we describe a way to achieve this. As a necessary condition for \mathcal{C}^s smooth transition functions, the union of the three boundary arcs in condition (iii) of Definition 3 has to be a \mathcal{C}^s smooth curve, as it is the image of a circular arc under a \mathcal{C}^s smooth mapping. By collecting charts and transition functions we obtain the atlas of the manifold which is associated with the triangular mesh M.

Definition 10 The triplet $\mathcal{A} = (C, \Phi_E, \Phi_F)$, where C is the set of all charts C^i and

$$\Phi_E = \{\phi_i^i : \{i, j\} \in E\} \text{ and } \Phi_F = \{\phi_{ik}^i : \{i, j, k\} \in F\}$$
(15)

are the sets of all edge and all face subchart parameterizations, will be called the C^s smooth parameterized atlas of the manifold, provided that all subchart parameterizations are valid.

The information about subcharts and transition functions is implicitly contained in the edge and face subchart parameterizations. The charts are considered to be mutually disjoint sets and they will be thought of as m_V different copies of the unit disk.

2.4 Manifold surface by blending

In order to define the spline manifold surface we define - for each chart - an embedding function, which is called the geometry function. We can then create blend surfaces that correspond to the overlapping subcharts.

Definition 11 For any $i \in V$, let $g^i : C^i \to \mathbb{R}^3$ be the associated **geometry** function. In addition, consider a scalar-valued function $\beta^i : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the following three conditions:

(i) β^i is C^s smooth, (ii) $\beta^i(x) > 0$ for $x \in \text{int } C^i$ and (iii) $\beta^i(x) = 0$ if $x \in \mathbb{R}^2 \setminus C^i$.

This function is called an *influence function*.

Now we are ready to define surface patches which are associated with vertices, edges and faces of the mesh by blending the geometry functions. Their collection forms the spline manifold surface.

Definition 12 We define patches for each vertex, each edge and each face of the given mesh.

(1) For any vertex with index $i \in V$, we call the mapping

$$\pi^{i}: \hat{C}^{i} \to \mathbb{R}^{d}: x \mapsto \boldsymbol{g}^{i}(x) \tag{16}$$

which is obtained by restricting the geometry function to the innermost part of the chart the **vertex patch** associated with the *i*-th vertex.

(2) For any edge with indices $e = \{i, j\} \in E$, let

$$\pi^{e}: \Box \to \mathbb{R}^{3}: x \mapsto \frac{\sum_{(p,q) \in \{(i,j), (j,i)\}} (\beta^{p} \circ \phi_{q}^{p})(x) \cdot (\boldsymbol{g}^{p} \circ \phi_{q}^{p})(x)}{\sum_{(p,q) \in \{(i,j), (j,i)\}} (\beta^{p} \circ \phi_{q}^{p})(x)}.$$
 (17)

This parameterization defines the **edge patch** associated with the edge e. (3) For any face with indices $f = \{i, j, k\} \in F$, let

$$\pi^{f}: \ \triangle \to \mathbb{R}^{3}: \ x \mapsto \frac{\sum_{\substack{(p,q,r) \in \{(i,j,k), \\ (j,k,i),(k,i,j)\}}} (\beta^{p} \circ \phi^{p}_{qr})(x) \cdot (\boldsymbol{g}^{p} \circ \phi^{p}_{qr})(x)}{\sum_{\substack{(p,q,r) \in \{(i,j,k), \\ (j,k,i),(k,i,j)\}}} (\beta^{p} \circ \phi^{p}_{qr})(x)}$$
(18)

This parameterization defines the **face patch** associated with the face f.

The collection of vertex, edge and face patches is said to be the **blending** manifold surface which is associated with the C^s smooth parameterized atlas \mathcal{A} and the geometry and influence functions.

Remark 13 If the subchart parameterizations ϕ_j^i and ϕ_{jk}^i , the influence functions β^i and the geometry functions \mathbf{g}^i are chosen as rational functions, then all patches of the blending manifold surfaces are rational, too.

Theorem 14 For any C^s smooth parameterized atlas $\mathcal{A} = (C, \Phi_E, \Phi_F)$ with associated geometry functions \mathbf{g}^i and influence functions β^i , we consider the collection of vertex patches, edge patches and face patches. Then any two neighboring patches meet with geometric continuity of order s in common points, provided that they are regular there.

For the proof of this theorem it suffices to observe that in a neighborhood of common points we can parameterize the union of two (or three) patches as a C^s smooth function over an open subset of one of the charts. Note that the definition of the parameterized atlas assumes that the transition functions are C^s smooth, hence such a smooth reparameterization can be found.

If the conditions of the theorem are satisfied, then the collection of face, edge and vertex patches is called a C^s smooth blending manifold surface. Note that – depending on the choice of the geometry functions – the edge, face and vertex patches may have singular points. For a generic choice of the geometry functions, these surface patches are all regular.

3 Construction of rational blending manifolds

This section describes a construction of a smooth blending manifold surface from a given triangular mesh. We generate the face and edge subchart parameterizations and choose the blending and geometry functions.

3.1 Parameterization of face subcharts

We consider a given triangular mesh M consisting of m_V vertices and m_F oriented triangles. In addition, we assume that a normal vector for each vertex is given. In many cases it can be estimated by fitting a plane to the triangular fan of the vertex. We assume that the orthogonal projection of the triangular fan of the vertex into the tangent plane (i.e., the plane through the vertex and perpendicular to the normal vector) is bijective¹.

All charts C^i will be chosen as circular disks with radius 1, centered at the origin. For each vertex *i*, we project the triangular fan of the vertex into the tangent plane of the vertex. The unit circle in the tangent plane is identified with the boundary ∂C^i of the chart, where the *x*-axis is (e.g.) aligned with the projection of the edge (i, n(i, 1)).

The intersection of the rays spanned by the projection of the edge $\{i, n(i, j)\}$ with the unit circle in the tangent plane defines the v(i) auxiliary points \mathbf{p}_{j}^{i} on the unit circle, where the lower index is counted modulo v(i), see Fig. 7. Next we compute the bisectors \mathbf{b}_{j}^{i} of the arcs from \mathbf{p}_{j}^{i} to \mathbf{p}_{j+1}^{i} . Finally, \mathbf{a}_{2j-1}^{i} is chosen as the point which divides the arc from \mathbf{b}_{j}^{i} to \mathbf{b}_{j-1}^{i} by the ratio 1 : 5, and \mathbf{a}_{2j}^{i} is chosen as the point which divides the arc from \mathbf{b}_{j}^{i} to \mathbf{b}_{j+1}^{i} by the ratio 1 : 5. See Remark 16 for a comment on the choice of this ratio.

¹ If this assumption is violated, then one can try to make the mesh smoother by applying a local averaging operator, or one should manually specify the quantities associated with the vertex.



Fig. 7. Left: Orthogonal projection of the triangular fan of a vertex into the tangent plane. Right: The points \mathbf{p}_{i}^{i} and \mathbf{a}_{i}^{i} and the layout of the subcharts.



Fig. 8. The control points of a face subchart.

Summing up, we obtain 2v(i) points $\mathbf{a}_1^i, \ldots, \mathbf{a}_{2v(i)}^i$ on the unit circle ∂C^i . The construction ensures that the arc lengths of the boundary arcs of the face and edge subcharts have approximately the ratio 1:2:1.

We choose the face subchart parameterization $\phi^i_{n(i,j-1),n(i,j)}$ as a planar rational Bézier triangle of degree two. The control points are chosen as follows:

- $\mathbf{b}_{200} = \mathbf{a}_{2j}^i, \ \mathbf{b}_{020} = \mathbf{a}_{2j-1}^i,$
- \mathbf{b}_{110} satisfies $\mathbf{b}_{110} \cdot \mathbf{b}_{200} = \mathbf{b}_{110} \cdot \mathbf{b}_{020} = 1$, i.e., it is the intersection of the circle tangents at \mathbf{b}_{200} and \mathbf{b}_{020} ,
- \mathbf{b}_{002} is chosen such that the triangle \mathbf{b}_{200} , \mathbf{b}_{020} , \mathbf{b}_{002} is equilateral,
- $\mathbf{b}_{011} = \frac{1}{2}(\mathbf{b}_{020} + \mathbf{b}_{002}), \ \mathbf{b}_{101} = \frac{1}{2}(\mathbf{b}_{200} + \mathbf{b}_{002}).$

The associated weights are

$$w_{200} = w_{020} = w_{002} = w_{011} = w_{101} = 1,$$

$$w_{110} = \cos \frac{1}{2} \angle (\mathbf{b}_{020}, \mathbf{0}, \mathbf{b}_{200}),$$
(19)

see Figure 8.

Any triangular patch of degree 2 can also be represented as a tensor-product patch of degree (2, 2), where one of the edges collapses into a singular point. If (u, v, w) are the barycentric parameters of the triangular patch (satisfying u + v + w = 1), then the reparameterization

$$\rho: \Box \to \triangle: (r, s) \mapsto (r, (1-r)s, (1-r)(1-s))$$

$$(20)$$

produces a biquadratic rational tensor-product patch whose edge r = 1 collapses into a singular point.

If the singular point is located at \mathbf{b}_{200} , then the 3 × 3 control points and weights of this degenerate patch can be generated simply by applying two-fold degree elevation to the control point \mathbf{b}_{200} , one-fold degree elevation to the two

control points \mathbf{b}_{101} , \mathbf{b}_{110} with associated weights and zero-fold degree elevation (i.e. just copying) to the three control points \mathbf{b}_{002} , \mathbf{b}_{011} , \mathbf{b}_{020} with associated weights. Since the last three control points describe a degree–elevated curve of degree 1 (a line), it is possible to reduce the degree with respect to s by one, giving a rational tensor-product patch of degree (1, 2) with the control points

$$\mathbf{c}_{02} = \mathbf{c}_{12} = \mathbf{b}_{200}, \mathbf{c}_{01} = \mathbf{b}_{110}, \ \mathbf{c}_{00} = \mathbf{b}_{020}, \mathbf{c}_{11} = \mathbf{b}_{101}, \ \mathbf{c}_{10} = \mathbf{b}_{002},$$
 (21)

where the weights are all equal to one, except for $w_{01} = w_{110}$.

3.2 Parameterization of edge subcharts

In order to construct the edge subchart parameterizations ϕ_j^i and ϕ_i^j , we consider the four neighboring face subchart parameterizations. They are reparameterized as tensor-product patches with singular points at the vertices that point away from the edge subcharts C_i^i and C_i^j , see Figure 9, left.

Once the edge subchart parameterizations are known, we have two parameterizations of the overlapping regions O_j^i and O_i^j , whose domain is the union of three (different copies of) unit squares. We choose the edge subchart parameterizations ϕ_j^i and ϕ_i^j such that these two parameterizations of the overlapping regions are \mathcal{C}^s smooth.

More precisely, the edge subchart parameterization ϕ_j^i (and analogously ϕ_i^j) has to satisfy the following two conditions:

- It has a \mathcal{C}^s smooth joint with the tensor-product patches $\phi_{jk}^i \circ \rho$ and $\phi_{lj}^i \circ \rho'$ along its edges E_4 and E_2 , respectively.
- Its boundary $\phi_i^i(E_3)$ is contained in the boundary ∂C^i .

This is achieved with the help of Möbius transformations.

Remark 15 A Möbius transformation is a special mapping of the plane into itself, where the plane is identified with the complex plane \mathbb{C} and closed by adding a single point ∞ at infinity. The mapping has the form

$$\mu: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}: z \mapsto \frac{az+b}{cz+d}$$
(22)

where $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. Möbius transformations are quadratic birational transformations that map circles onto circles, where lines are seen as circles with infinite radius. The inverse of a Möbius transformation is again a Möbius transformation. A Möbius transformation is uniquely determined by prescribing three different images for three different points.



Fig. 9. Left: The construction of C^s smooth transition functions Φ^{ij} using parameterization of the face subcharts by degenerate tensor-product patches. The singular points are indicated by circles. Right: The transformation μ is determined by the values at three points. To ensure that μ maps the unit circle onto a straight line, we require that $\mu(z_3) = \infty$.

Let μ be the Möbius transformation that maps the vertices \mathbf{z}_1 , \mathbf{z}_2 of the subcharts C_{lj}^i and C_{jk}^i that point away from the edge subcharts C_j^i into -1 and +1. In addition, a third point \mathbf{z}_3 on the unit circle is mapped to ∞ , see Fig. 9, right. For instance, this point can be chosen as the intersection point of the bisector of \mathbf{z}_1 , \mathbf{z}_2 with the unit circle which is farther away than the other intersection point. Due to $\mu(z_3) = \infty$, the unit circle is mapped into the real axis, and the interior of the disk is mapped to the upper half of the complex plane.

The edge subchart parameterization ϕ_j^i is now found by the following construction (see Fig. 10):

(1) The face subchart parameterizations ϕ_{lj}^i and ϕ_{jk}^i are represented as degenerate tensor-product patches and they are composed with the Möbius transformation μ . This gives two rational tensor-product patches

$$\zeta_{lj}^{i} = \mu \circ \phi_{lj}^{i} \circ \rho' \quad \text{and} \quad \zeta_{jk}^{i} = \mu \circ \phi_{jk}^{i} \circ \rho \tag{23}$$

of degree (2, 4) that parameterize the images $\mu(C_{lj}^i)$ and $\mu(C_{jk}^i)$ of the face subcharts under the Möbius transformation μ .

- (2) We now create a tensor-product patch ξ_j^i that has a \mathcal{C}^s smooth joint with ζ_{jk}^i and ζ_{lj}^i along its edges E_4 and E_2 and whose boundary curve $\xi_j^i(E_3)$ is contained in the real axis. This rational patch can be chosen as a rational tensor-product patch of degree (2, 2s + 1). On either side the first s rows of control points can immediately be found with the help of the control points of ζ_{lj}^i and ζ_{jk}^i . The edge E_3 is then automatically mapped into the real axis.
- (3) We apply the inverse Möbius transformation in order to get the desired



Fig. 10. (a) The two face subcharts C_{lj}^i and C_{jk}^i are parameterized as degenerated quadrangular patches; the singular point is marked. (b) The Möbius transformation μ maps C_{lj}^i and C_{jk}^i into the upper half plane; (c) ξ_j^i is parameterized satisfying the \mathcal{C}^s continuity conditions; (d) μ^{-1} maps ξ_j^i back into the unit ball.

edge subchart parameterization

$$\phi_j^i = \mu^{-1} \circ \xi_j^i. \tag{24}$$

This gives a rational tensor product patch of degree (4, 4s + 2) that possesses the desired properties.

The edge subchart parameterization ϕ_i^j can be constructed similarly. Fig. 11 shows an example of an edge subchart parameterization.

Remark 16 According to our experience, which is supported by various experimental results, the method described in sections 3.1 and 3.2 produces regular parameterizations. It also ensures that the interiors of the subcharts are pairwise disjoint, also for vertices of valence 3 and for highly non-uniform distributions of the points \mathbf{p}_{j}^{i} . Currently we do not have a theoretical guarantee for this. However, in case of any problems, it would be possible to use a lower ratio than 1 : 5 for computing the vertices \mathbf{a}_{k}^{i} of the subcharts. In the



Fig. 11. The four steps for constructing an edge subchart parameterization, where the chosen degree of continuity is s = 2. The parameter lines show the singular points for the face subcharts after the reparameterization as degenerated quadrangular patches.



Fig. 12. The innermost part for the chart C^i . In this case v(i) = 5 Bézier triangles are needed.

limit, the face subcharts shrink to points, and the innermost boundaries of the edge tend to arcs of the circle boundary.

3.3 Innermost parts, geometry functions and influence functions

The remaining (or innermost) part \hat{C}^i of each chart can be parameterized as a trimmed tensor-product patch of degree (1,1). Alternatively, it can be parameterized by v(i) rational Bézier triangles of degree 4s + 2, simply by choosing a vertex of the origin and connecting it with the innermost vertices $C_{jk}^i \cap \hat{C}^i$, $\{i, j, k\} \subset F$, of the face subcharts. See Figure 12 for an example.

The geometry functions $\mathbf{g}_i: C^i \to \mathbb{R}^3$ can be chosen in various ways. In our

implementation, they are automatically generated from the mesh, taking the form

$$\mathbf{g}_{i}(u,v) = \gamma \cdot \left((1-\lambda) \cdot \mathbf{t}_{i}(u,v) + \lambda \cdot q_{i}(u,v) \cdot \mathbf{n}_{i}\right)$$
(25)

where \mathbf{t}_i is a linear parameterization of the tangent plane at the vertex *i* of the mesh, \mathbf{n}_i is the normal vector and $q_i(u, v)$ is a quadratic polynomial. This quadratic polynomial is automatically computed by fitting the neighbors of the vertex to the geometry function.

The parameter γ is a *shrinking factor*, that controls the size of the embedding of the chart. The parameter λ is a *flatness factor*. It controls the flatness of the chart embedding. The parameters γ and λ control the distance between the manifold surface and the control mesh. For small values of γ , the surface is closer to the mesh.

Unless otherwise specified, we have used $\lambda = 0.5$ in all our examples. The factor γ was chosen depending on the valence of the vertex, varying between 0.7 for valence 3 and 0.3 for valence 10.

We choose influence function β^i as

$$\beta^{i}(u,v) = (1 - u^{2} - v^{2})_{+}^{s+1}$$
(26)

where $(x)_{+} = \frac{1}{2}(x + |x|)$ and s is the desired order of smoothness.

Remark 17 The choice of the geometry functions depends on the data which is available. E.g., if the triangular mesh is obtained by discretizing an implicitly defined surface, then one should choose the geometry functions as approximate parameterizations of the neighborhoods of the vertices (cf. Wurm et al., 1997). On the other hand, for other applications it is useful to consider geometry functions which add details and features (cf. Yoon, 2006). If the geometry functions are chosen as polynomials of degree higher than two, then the manifold surface becomes locally more flexible.

4 Examples

This section presents some examples.

Example 1. This surface (see Figure 13) is generated from a triangular mesh describing a double torus. The mesh has 284 faces and 140 vertices. The



Fig. 13. A triangular mesh describing a double torus (left) and the associated smooth blending manifold surface (center and right). The surface was rendered using 17,890 triangles.

associated blending C^2 manifold surface is shown in the right picture. The yellow, blue and red regions correspond to the vertex, edge and face patches, respectively.

Example 2. Fig. 14 compares the adaptive approach (top row) and a nonadaptive approach (bottom), where the subcharts depend only on the valence of the vertices. In the left column, the model is rendered by patch type. Although for some vertices the result is quite similar, at some vertices the adaptive approach produces a more natural (less twisted) result. In the right column, we zoomed into the part of the surface marked by the green rectangle. It is clearly visible that the adaptive approach leads to a more desirable result than the one obtained by the non-adaptive method.

Example 3. Fig. 15 shows several C^2 -smooth surfaces obtained by our construction. The surface in Fig. 15 (a) describes a bunny generated from a triangular mesh with 253 vertices and 502 faces: the manifold has been rendered by patch type in order to show that the method is able to handle charts with any valence. Moreover, it is also possible to choose different sizes / shape parameters for the geometry functions, in order to get sensible results for more complicated parts of the mesh, such as the ears of the bunny. The model in Fig. 15 (b) is a toroidal surface generated from a mesh with 48 vertices and 96 faces. The example in Fig. 15 (c) describes a surface associated with a mesh consisting of 10 vertices and 16 faces.

Example 4. Fig. 16 shows the influence of the parameters λ and γ which control the geometry functions, see (25). The control mesh of the star-shaped polyhedron consists of 18 vertices and 32 faces. Depending on the choice of the parameters, the relative size of the edge and face subcharts changes. The models in the first row have a low flatness factor λ . In the second row λ has



Fig. 14. Comparison between adaptive (top row) and non-adaptive method (bottom row) for the double torus model of Fig. 13.



Fig. 15. Various C^2 smooth surfaces which demonstrate the possibilities of the presented construction. The surfaces were rendered using 31,626, 6,048, and 9,072 triangles, respectively.



Fig. 16. C^2 manifold surface obtained from a star–shaped polyhedron, for different values of the shape parameters λ and γ controlling the geometry functions. The surface was rendered using 14,336 triangles.

been increased and each geometry function gets closer to the tangent plane at the vertex of the mesh.

Example 5. The final example demonstrates the effects of different smoothness. We created a sphere-like manifold surface from an icosahedral mesh (12 vertices and 20 faces) for different values of the smoothness s. In the top row of Fig. 4 the C^2 manifold is rendered in different types: by chart type, using a single color with shading effects and showing the reflection of a checkerboard pattern. The three remaining figures zoom in the blue rectangle. In the case of surfaces which are only C^1 manifolds (left), the reflection of the checkerboard pattern creates curves with tangent discontinuities. In the C^2 and C^3 case (center and right), these curves are much smoother.



Fig. 17. Top row: C^2 smooth surface obtained from a icosahedron and the reflection of a checkerboard pattern in it. Bottom row: Detail of the reflected checkerboard pattern (see blue box in the top right picture) for different degrees of smoothness. The surfaces were rendered using 3500 triangles.

Remark 18 The computation of the manifold surfaces presented in the paper has been implemented using Maple10 and the figures have been generated with the help of PovRay. The computational time depends linearly on the number of vertices m_V of the original mesh. The chart generation and the computation of the geometry function takes approximately 5 sec per vertex. The complete algorithm (charts and geometry functions generation, embedding, blending and triangulation of the surfaces) is evaluated in approximately 14 sec per vertex. In the presented examples, we triangulate each quadrangular surface and each triangular surface in 50 and 25 triangles, respectively.

5 Conclusions and future work

We presented a novel construction of a rational manifold surface with arbitrary order s of smoothness from a given triangular mesh. The given triangular mesh is used both to guide the geometry functions and to define the connectivity of the charts. The transition functions are obtained via subchart parameterizations. The manifold surface can be described as a collection of quadrangular and triangular (untrimmed) rational surface patches.

As an advantage with respect to subdivision surfaces, the use of manifold surfaces provides an explicit parameterization of the surface. Moreover, it is possible to get any order of smoothness and there are no difficulties associated with "extraordinary" vertices. Our construction can generate surfaces from triangular meshes, while other constructions need quadrilateral meshes (Ying and Zorin, 2004) or 4-valent meshes (Grimm and Hughes, 1995). For our construction, the point-wise evaluation requires only rational operations, while (Ying and Zorin, 2004) need more complicated functions.

In our construction, the subcharts and transition functions can be adapted to the geometry of the given triangular mesh. This is an important difference to the method of Ying and Zorin (2004), where the transition functions and charts depend solely on the connectivity, but not on the geometry of the given quadrilateral mesh.

The construction provides many possibilities for additional investigations, e.g., concerning the optimal choice of charts (which do not need to be circular) and geometry functions. In particular, techniques for optimizing the surfaces with respect to fairness measures should be of some interest. In addition, future work will concentrate on two issues. First, we plan to address boundary conditions and sharp features (edges) that may be present in a given object. Second, we plan to investigate other manifold constructions which are based on subchart parameterization. In particular we will investigate construction methods for surfaces of relatively low degree.

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