Curves and surfaces represented by polynomial support functions

Dedicated to André Galligo on the occasion of his 60th birthday.

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Abstract

This paper studies shapes (curves and surfaces) which can be described by (piecewise) polynomial support functions. The class of these shapes is closed under convolutions, offsetting, rotations and translations. We give a geometric discussion of these shapes and present methods for the approximation of general curves and surfaces by them. Based on the rich theory of spherical spline functions, this leads to computational techniques for rational curves and surfaces with rational offsets, which can deal with shapes without inflections/parabolic points.

Key words: polynomial support function, approximation by spherical splines, offset surfaces, convolutions.

1 Introduction

Due to their importance for various applications, *offset* curves and surfaces have been subject of intensive research in Computer Aided Design (CAD). Offsetting is closely related to the notion of the *convolution* of two surfaces, which contains offsetting as a special case (convolution with a sphere).

The class of (piecewise) rational curves and surfaces (i.e., NURBS), which is frequently used in CAD, is not closed under offsetting and convolutions. For this reason, several approximate techniques have been developed [4,5,15]. These techniques require a careful control of the approximation error. In particular, each offset curve or surface has to be approximated separately.

On the other hand, it is possible to identify subsets of the space of rational curves and surfaces which are closed under offsetting, or even under the (more general) convolution operator. In the curve case, this led first to the interesting class of polynomial Pythagorean–hodograph (PH) curves, see [6] and the references cited therein. This class of curves is now fairly well understood, and various computational techniques for generating them are available.

This approach has later been extended to the surface case, by introducing the class of rational PH curves and Pythagorean–normal vector (PN) surfaces [17,16]. This class has been defined by using a very elegant construction, which provides a dual control structure: Starting from a dual parametric representation of the unit circle/sphere, the dual control structure of a rational PH curve/PN surface is obtained simply by applying parallel displacements to the control lines/planes.

In practice, however, it turned out that it is very difficult to use this dual control structure for curve and surface design [20]. This motivated the investigation of alternative representations, which may even deal with the more general operation of convolution [13,19].

In order to deal with offsets and convolutions, the present paper studies the support function representation of curves and surfaces. Roughly speaking, a curve/surface is described by the distance of its tangent planes to the origin of the coordinate system, which is the used to define a function on the unit circle/unit sphere. This representation is one of the classical tools in the field of convex geometry, see e.g. [3,11,12]. Its application to problems in Computer Aided Design can be traced back to a classical paper of Sabin [18].

In order to use this representation for geometric design, we are particularly interested in the case of (piecewise) polynomial support functions. By using functions of this type, it is possible to apply the well–developed theory of spline functions on the sphere to this case [1].

The remainder of the paper is organized as follows. After recalling some notions from differential geometry in Section 2, the third sections shows how to describe shapes by their support function. We introduce the linear space of quasi–convex shapes and discuss smoothness of the surfaces and norms of associated operators. Section 5 discusses the case of *polynomial* support functions. It is shown that any shape with a polynomial support function can be obtained as the convolution of finitely many elementary shapes, which can be derived from certain hypocycloids [1]. Section 6 is devoted to computational techniques for approximating general support functions by (piecewise) polynomial ones. Finally, we conclude this paper.

[1] In the curve case, related results were already noted in [8].

2 Some notions from differential geometry

In this section we recall some fundamental notions from differential geometry: tangent spaces, intrinsic gradients and Hessians of functions defined on manifolds, covariant derivatives, and differentials of mappings. We will present all these notions in the case of manifolds which are *embedded hypersurfaces*, where they can be obtained via projection into the tangent space.

We consider a smooth oriented d–dimensional manifold M ($d = 1$: a curve, $d = 2$: a surface) which is embedded into the $d+1$–dimensional space $\mathbb{R}^{d+1}$. The latter space is equipped with the usual inner product (denoted by '$\cdot$'). In particular, we are interested in the case of the unit sphere ($d = 1$: circle) S^d. In order to avoid double indices, we will omit the dimension d, writing S instead of S^d.

2.1 Tangent spaces and gradients

For any point $\mathbf{p} \in M$ we have an associated *unit normal vector* $\mathbf{n_p}$ which defines the *tangent space*

$$T_\mathbf{p}M = \{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{n_p} = 0\} \subset \mathbb{R}^{d+1}, \tag{1}$$

along with the orthogonal projection

$$\pi_\mathbf{p} : \ \mathbb{R}^{d+1} \to T_\mathbf{p}M : \ \mathbf{x} \mapsto \mathbf{x} - (\mathbf{x} \cdot \mathbf{n_p})\mathbf{n_p}. \tag{2}$$

Let $h \in C^1(\mathbb{R}^{d+1}, \mathbb{R})$ be a real–valued function. The restriction of h to M defines a C^1 function on the manifold M.

For any vector $\mathbf{v} \in T_\mathbf{p}M$ we define the directional derivative

$$\mathcal{D}_\mathbf{p}(\mathbf{v})h = (\mathbf{v} \cdot \nabla)h \,\Big|_\mathbf{p}\,, \tag{3}$$

where ∇ is the usual nabla operator in $\mathbb{R}^{d+1}$, which is used like a column vector. Moreover, the vector

$$\nabla_M h \,\Big|_\mathbf{p} = \pi_\mathbf{p}(\nabla h \,\Big|_\mathbf{p}), \tag{4}$$

is called the *gradient* of h with respect to the manifold M. Observe that if $\mathbf{v} \in T_\mathbf{p}M$ then $\mathcal{D}_\mathbf{p}(\mathbf{v})h = (\mathbf{v} \cdot \nabla_M)h \,\Big|_\mathbf{p}$. The directional derivatives and the gradient of a function h with respect to a manifold M are fully determined by the restriction of h to M.

2.2 Covariant derivatives and Hessians

The restriction of a vector–valued function $\mathbf{w} \in C^1(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})$ to M defines a *vector field* on M, provided that $\mathbf{w}(\mathbf{p}) \in T_\mathbf{p}M$ holds for all points $\mathbf{p} \in M$. For any point $\mathbf{p} \in M$ and tangent vector $\mathbf{v} \in T_{\mathbf{p}}M$, the vector

$$\mathcal{D}_{\mathbf{p}}(\mathbf{v})\mathbf{w} = \pi_{\mathbf{p}}(\,(\mathbf{v} \cdot \nabla)\mathbf{w}\,\Big|_{\mathbf{p}}) \tag{5}$$

is called the *covariant directional derivative* of the vector field $\mathbf{w}$ with respect to the direction $\mathbf{v}$ at $\mathbf{p}$. Again, $\mathcal{D}_{\mathbf{p}}(\mathbf{v})\mathbf{w}$ is fully determined by the restriction of $\mathbf{w}$ to M.

Let $h \in C^2(\mathbb{R}^{d+1}, \mathbb{R})$ be again a real–valued function. The linear mapping

$$\mathrm{Hess}_M\Big|_{\mathbf{p}} : T_{\mathbf{p}}M \to T_{\mathbf{p}}M : \mathbf{v} \mapsto \mathcal{D}_{\mathbf{p}}(\mathbf{v})(\nabla_M h) \tag{6}$$

is called the *Hessian* of the function h with respect to the manifold M at the point $\mathbf{p}$. Once more, the Hessian of a function h with respect to a manifold M is fully determined by the restriction of h to M.

2.3 The differential of a mapping between manifolds

We consider a function $\mathbf{x} \in C^1(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})$, which is now seen as a mapping of $\mathbb{R}^{d+1}$ to itself, with the Jacobian

$$J(\mathbf{x})\Big|_{\mathbf{p}} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} : \mathbf{v} \mapsto (\mathbf{v} \cdot \nabla)\mathbf{x}\Big|_{\mathbf{p}} \tag{7}$$

We assume that the image of the manifold M is contained in another smooth manifold N. Then, for any point $\mathbf{p} \in M$, the restriction of the Jacobian to $T_{\mathbf{p}}M$ maps the tangent space of M into the tangent space of N at $\mathbf{x}(\mathbf{p})$. This linear mapping

$$\mathrm{d}\mathbf{x}\Big|_{\mathbf{p}} : T_{\mathbf{p}}M \to TN_{\mathbf{x}(\mathbf{p})} : \mathbf{v} \mapsto (\mathbf{v} \cdot \nabla)\mathbf{x}\Big|_{\mathbf{p}} \tag{8}$$

is called the *differential* of the mapping $\mathbf{x} : M \to N$ at $\mathbf{p}$. The differential depends solely on the restriction of $\mathbf{x}$ to M.

2.4 The Gauss map and the Weingarten map

Recall that the *Gauss map* $\mathbf{G}$ of an embedded hypersurface assigns to a point the associated unit normal,

$$\mathbf{G} : M \to S : \mathbf{p} \mapsto \mathbf{n}_{\mathbf{p}}. \tag{9}$$

All properties concerning the curvature of M at a point $\mathbf{p}$ can be derived from the Weingarten map $W = -\mathrm{d}\mathbf{G}$. Since the tangent spaces of M at $\mathbf{p}$ and of S at $\mathbf{n}_{\mathbf{p}}$ are identical, the map W is a linear map of $T_{\mathbf{p}}M$ into itself. The eigenvectors and eigenvalues of the Weingarten map are the principal directions and principal curvatures, respectively, and the determinant of W is the Gaussian curvature of M at $\mathbf{p}$. The case of curves, along with an application to mechanical design, has been studied in [9,10].

3 Spherical harmonics

The first two sections summarize several results about spherical harmonics and Fourier analysis. The third section defines a special basis of spherical polynomials of bounded degree, which will be needed later to discuss shapes with polynomial support functions. See [11] for more information on harmonic analysis and its application in geometry.

3.1 The Laplace-Beltrami operator on S

Let $h \in C^2(\mathbb{R}^{d+1}, \mathbb{R})$ be a real–valued function. The *Laplace-Beltrami operator* on S is defined by

$$\Delta_S h \Big|_{\mathbf{p}} = \left(\Delta h - \frac{\partial^2 h}{\partial \mathbf{n_p}^2} - d \frac{\partial h}{\partial \mathbf{n_p}} \right) \Big|_{\mathbf{p}}, \tag{10}$$

where Δ is the usual Laplace operator in $\mathbb{R}^{d+1}$ and $\partial^i / \partial \mathbf{n}^i_{\mathbf{p}}$, $i = 1, 2$, denotes the first and is the second derivative along the normal line of S at $\mathbf{p}$, respectively. The value of the Laplace-Beltrami operator depends solely on the restriction of h to S.

A homogeneous polynomial $p(\mathbf{x}) = p(x_0, \ldots, x_d)$ of degree k is called *harmonic* if $\Delta p = 0$, where Δ is the Laplace operator in $\mathbb{R}^{d+1}$. We denote the spaces of homogeneous and harmonic polynomials of degree k by $\mathcal{P}^k$ and $\mathcal{H}^k$, respectively. The Laplace operator maps $\mathcal{P}^k$ onto $\mathcal{P}^{k-2}$. As $\mathcal{H}^k$ is the kernel,

$$\dim \mathcal{P}^k = \dim \mathcal{H}^k + \dim \mathcal{P}^{k-2} = \cdots = \dim \mathcal{H}^k + \dim \mathcal{H}^{k-2} + \cdots + \dim \mathcal{H}^\sigma, \tag{11}$$

where $\sigma = 0$ if k is even and $\sigma = 1$ if k is odd. Since $\dim \mathcal{P}^k = \binom{d+k}{k}$ we obtain

$$\dim \mathcal{H}^k = \binom{d+k}{k} - \binom{d+k-2}{k-2} = \frac{2k+d-1}{k+d-1} \binom{k+d-1}{d-1}. \tag{12}$$

The restriction of a harmonic polynomial of degree k to the unit sphere S is called a *spherical harmonic* of degree k. A homogeneous polynomial p of degree k is determined by its value on the unit sphere S. For $p \in \mathcal{P}^k$, $r \in \mathbb{R}_+$ and $\mathbf{n} \in S$ one obtains

$$p(r\mathbf{n}) = r^k p(\mathbf{n}), \qquad \frac{\partial p}{\partial r} \Big|_{\mathbf{n}} = kp(\mathbf{n}) \qquad \text{and} \qquad \frac{\partial^2 p}{\partial r^2} \Big|_{\mathbf{n}} = k(k-1)p(\mathbf{n}). \tag{13}$$

If $p \in \mathcal{H}^k$ is a spherical harmonic of degree k, then it is also an eigenfunction of Δ_S with the eigenvalue $-k(k+d-1)p$,

$$\Delta_S \, p = -k(k-1)p - dkp = -k(k+d-1)p. \tag{14}$$

see (10). These spherical harmonics are the only eigenfunctions.

3.2 Fourier analysis on S

From now on we consider the elements of the spaces $\mathcal{P}^k$ and $\mathcal{H}^k$ as *functions* on S. Note that the degree of $p \in \mathcal{P}^k$ is not unique, as $p(\mathbf{n}) = \|\mathbf{n}\|^2 p(\mathbf{n})$ for $\mathbf{n} \in S$.

On the other hand, for any two polynomials $p \in \mathcal{P}^k$ and $q \in \mathcal{P}^\ell$ which agree on S, we get $q(\mathbf{x}) = \|\mathbf{x}\|^{\ell-k} p(\mathbf{x})$, hence $\ell - k$ has to be even. Furthermore, if $p \in \mathcal{H}^k$, then $\Delta \|\mathbf{x}\|^{2\ell} p(\mathbf{x}) = \ell \|\mathbf{x}\|^{2\ell-2} p(\mathbf{x}) \neq 0$ unless $\ell = 0$, so the extension of a spherical harmonic to a harmonic polynomial is unique. Consequently, $\mathcal{H}^k \cap \mathcal{H}^\ell = \{0\}$ if $k \neq \ell$ and we obtain from (11)

$$\mathcal{P}^k = \mathcal{H}^k \oplus \mathcal{H}^{k-2} \oplus \cdots \oplus \mathcal{H}^\sigma, \tag{15}$$

where $\sigma = 0$ or 1. (Recall the the polynomials are considered as functions on S.)

Let $\mathcal{P}^{\leq k}$ and $\mathcal{P}$ denote the space of polynomials of degree $\leq k$ and the space of all polynomials, respectively. They can be expressed as direct sums of spaces of spherical harmonics,

$$\mathcal{P}^{\leq k} = \mathcal{P}^k \oplus \mathcal{P}^{k-1} = \bigoplus_{\ell=0}^{k} \mathcal{H}^\ell \qquad \text{and} \qquad \mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{H}^k. \tag{16}$$

As Δ_S is a self-adjoint operator, the spaces of harmonic functions satisfy $\mathcal{H}^k \perp \mathcal{H}^\ell$ with respect to the inner product $\langle \cdot, \cdot \rangle$ of $L^2(S)$. In harmonic analysis, one now chooses an orthonormal basis for each of the spaces $\mathcal{H}^k$. By collecting these bases one obtains an orthonormal basis $\psi_1, \psi_2, \ldots$ of $\mathcal{P}$. The inner products $c_k = \langle f, \psi_k \rangle$ are then called the *Fourier coefficients* of a given function $f \in L^2(S)$, and $\sum_{k=1}^{\infty} c_k \psi_k$ is called the *Fourier series* of f, both with respect to the given orthonormal basis.

3.3 A basis of polynomials of bounded degree

Instead of choosing a particular orthonormal basis of the spaces $\mathcal{H}^k$, we now consider expansions of the form

$$p = \sum_{j=0}^{k} p_j, \quad p_j \in \mathcal{P}^{\leq j}. \tag{17}$$

where $p \in \mathcal{P}^{\leq k}$. Clearly, a Fourier series is a particularly simple way to obtain an expansion of this form. We define the polynomials

$$P_j(x) = \sum_{\ell=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^\ell \binom{j}{2\ell} x^{j-2\ell} (1-x^2)^\ell, \tag{18}$$

which are obtained by expressing $\cos(j\theta)$ in $x = \cos\theta$. Later, it will be shown that these polynomials correspond to particularly simple geometric objects.

Lemma 1 *There exists a basis of $\mathcal{P}^{\leq k}$ which consists of polynomials of the form*

$$p_{\mathbf{a},\ell}(\mathbf{x}) = P_\ell(\mathbf{a}\cdot\mathbf{x}) = P_\ell(a_0x_0 + \cdots + a_dx_d), \quad \ell = 0,\cdots,k, \quad \mathbf{a}\in S, \tag{19}$$

where $\dim\mathcal{H}^\ell = \frac{2\ell+d-1}{\ell+d-1}\binom{\ell+d-1}{d-1}$ *different polynomials of degree ℓ (defined by different points $\mathbf{a}\in S$) are present.*

Proof. First we show that the polynomials (19) span the space $\mathcal{P}^{\leq k}$. As the degree of the polynomial P_k is k we have that the monomial x^k can be written as a linear combination of the polynomials $P_0,\cdots,P_k$. That means that we for $\mathbf{a}\in S$ can write $(\mathbf{a}\cdot\mathbf{x})^k$ as a linear combination of the polynomials $p_{\mathbf{a},0},\cdots,p_{\mathbf{a},k}$. As $(c\mathbf{a}\cdot\mathbf{x})^k = c^k(\mathbf{a}\cdot\mathbf{x})^k$ we can write $(c\mathbf{a}\cdot\mathbf{x})^k$ as a linear combination of the polynomials $p_{\mathbf{a},0},\cdots,p_{\mathbf{a},k}$. Finally,

$$\mathbf{x}^{\mathbf{k}} = \frac{1}{|\mathbf{k}|!}\sum_{\mathbf{l}\leq\mathbf{k}}\binom{\mathbf{k}}{\mathbf{l}}(\mathbf{l}\cdot\mathbf{x})^{|\mathbf{k}|},$$

where $\mathbf{k}$ and $\mathbf{l}$ are multi indices, $|\mathbf{k}| = k_0+\cdots+k_d$, $\mathbf{l}\leq\mathbf{k}$ if $l_0\leq k_0\wedge\cdots\wedge l_d\leq k_d$, $\binom{\mathbf{k}}{\mathbf{l}} = \binom{k_0}{l_0}\cdots\binom{k_d}{l_d}$, and $\mathbf{x}^{\mathbf{k}} = x_0^{k_0}\cdots x_d^{k_d}$. So any polynomial $\sum_{|\mathbf{k}|\leq k} c_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$ can be written as a linear combination of the polynomials (19).

The Lemma is now shown by induction on k. If $k=0$ then the it is obviously true. So assume we have a basis for$\mathcal{P}^{\leq k-1}$ of the required type. We can supplement this basis to a basis for $\mathcal{P}^{\leq k}$ with polynomials of the type (19), as the new members of the basis are in $\mathcal{P}^{\leq k}\setminus\mathcal{P}^{\leq k-1}$ we have $\ell = k$ and by (16) the number of new elements is $\dim\mathcal{H}^k$. $\square$

4 Defining shapes by their support function

In this section we introduce the support function representation of hypersurfaces and study its basic properties.

4.1 The envelope operator

From now on we often consider the unit sphere S as a d–dimensional manifold. Its points will simply be denoted by $\mathbf{n}$, since they coincide with the normals.

Definition 2 *Let $U\subseteq S$ be an open subset of the n dimensional unit sphere*[2] *and $h\in C^1(U,\mathbb{R})$ be the* **support function**. *Let $\mathbf{x}_h\in C^0(U,\mathbb{R}^{d+1})$ be defined as*

$$\mathbf{x}_h : \mathbf{n}\mapsto\mathbf{x}_h(\mathbf{n}) = h(\mathbf{n})\mathbf{n} + \nabla_S h\Big|_{\mathbf{n}}. \tag{20}$$

[2] A set $U\subseteq S$ is said to be open in S if there exists an open subset $\tilde{U}\subset\mathbb{R}^{d+1}$ such that $U = S\cap\tilde{U}$.

The linear operator

$$\mathcal{E} : C^1(U,\mathbb{R}) \to C^0(U,\mathbb{R}^{d+1}) : h \mapsto \mathbf{x}_h \tag{21}$$

is called the **envelope operator**.

Recall that we consider the unit sphere S as an *embedded* manifold in $\mathbb{R}^{d+1}$. Hence, the gradient $\nabla_S h$ at $\mathbf{n}$ is contained in $T_\mathbf{n}S \subset \mathbb{R}^{d+1}$, and $\mathbf{n} \in S \subset \mathbb{R}^{d+1}$.

The geometrical meaning of the formula (20) is as follows.

Proposition 3 *The vector–valued function $\mathbf{x}_h$ parameterizes the envelope of the family of the hyperplanes*

$$T_\mathbf{n} = \{\mathbf{x} : \mathbf{x}\cdot\mathbf{n} = h(\mathbf{n})\}, \quad \mathbf{n} \in U \subseteq S, \tag{22}$$

with normal vector $\mathbf{n}$ and distance $h(\mathbf{n})$ to the origin.

Proof. For any point $\mathbf{x} \in \mathbb{R}^{d+1}$ we consider the function $f_\mathbf{x} : S \to \mathbb{R}$

$$f_\mathbf{x} : \mathbf{n} \mapsto \mathbf{n}\cdot\mathbf{x} - h(\mathbf{n}). \tag{23}$$

If a point $\mathbf{x}$ belongs to the envelope and corresponds to a certain point $\mathbf{n}_0 \in S$, then it satisfies

$$f_\mathbf{x}(\mathbf{n}_0) = \mathbf{n}_0\cdot\mathbf{x} - h(\mathbf{n}_0) = 0 \quad \text{and} \quad \forall \mathbf{v} \in T_{\mathbf{n}_0}S:\ \mathcal{D}_{\mathbf{n}_0}(\mathbf{v})f_\mathbf{x} = 0. \tag{24}$$

A short computation leads to

$$D_{\mathbf{n}_0}(\mathbf{v})f_\mathbf{x} = ((\mathbf{v}\cdot\nabla)\mathbf{n})\cdot\mathbf{x} - (\mathbf{v}\cdot\nabla)h\,\Big|_{\mathbf{n}_0} = \mathbf{v}\cdot\mathbf{x} - (\mathbf{v}\cdot\nabla)h\,\Big|_{\mathbf{n}_0}, \tag{25}$$

since $(\mathbf{v}\cdot\nabla)\mathbf{n} = \mathbf{v}$. Consequently, after choosing a basis of $T_{\mathbf{n}_0}S$, we obtain from (24) a regular system of linear equations for $\mathbf{x}$, which has a unique solution. On the other hand, the point $\mathbf{x}_h(\mathbf{n}_0)$ fulfills the equations (24). $\square$

Proposition 4 *Let $h \in C^2(U,\mathbb{R})$, where $U \subseteq S$ is an open subset of the unit sphere. A point $\mathbf{n} \in U$ is called a regular point for the vector–valued function $\mathbf{x}_h$ if*

$$\det(\mathrm{Hess}_S h + hI)\,\Big|_\mathbf{n} \neq 0, \tag{26}$$

where I is the identity on $T_\mathbf{n}S$. The vector–valued function $\mathbf{x}_h$ is a regular parameterization, if and only if all points $\mathbf{n} \in U$ are regular. If this assumption is satisfied, then the tangent spaces of S at $\mathbf{n} \in U$ and of $M = \mathbf{x}_h(U)$ at $\mathbf{x}_h(\mathbf{n})$ are identical, the differential of $\mathbf{x}_h$ satisfies

$$\mathrm{d}\mathbf{x}_h = \mathrm{Hess}_S h + hI, \tag{27}$$

and $-(\mathrm{d}\mathbf{x}_h)^{-1}$ is the Weingarten map of M.

Proof. A short computation confirms that for any $\mathbf{v} \in T_{\mathbf{n}}S$

$$
\begin{aligned}
&(\mathrm{d}\mathbf{x}_h - hI - \mathrm{Hess}_S h)\Big|_{\mathbf{n}}(\mathbf{v}) \\
&= (\mathbf{v}\cdot\nabla)(h\mathbf{n}) + \underline{(\mathbf{v}\cdot\nabla)(\nabla_S h)} - h\mathbf{v} - \underline{(\mathbf{v}\cdot\nabla)(\nabla_S h)} + \{[(\mathbf{v}\cdot\nabla)(\nabla_S h)]\cdot\mathbf{n}\}\mathbf{n}\Big|_{\mathbf{n}} \\
&= (\mathbf{v}\cdot\nabla h)\mathbf{n} + \underline{h[\overbrace{(\mathbf{v}\cdot\nabla)\mathbf{n}}^{=\mathbf{v}}]} - \underline{h\mathbf{v}} + (\{(\mathbf{v}\cdot\nabla)[\overbrace{\nabla h - (\mathbf{n}\cdot\nabla h)\mathbf{n}}^{=\nabla_S h}]\}\cdot\mathbf{n})\mathbf{n}\Big|_{\mathbf{n}} \\
&= \underline{(\mathbf{v}\cdot\nabla h)\mathbf{n}} + \underline{\underline{\{[(\mathbf{v}\cdot\nabla)(\nabla h)]\cdot\mathbf{n}\}\mathbf{n}}} - \underline{\{[\overbrace{(\mathbf{v}\cdot\nabla)\mathbf{n}}^{=\mathbf{v}}]\cdot\nabla h\}(\overbrace{\mathbf{n}\cdot\mathbf{n}}^{=1})\mathbf{n}} \\
&\qquad - \underline{\underline{\{\mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\nabla h]\}(\overbrace{\mathbf{n}\cdot\mathbf{n}}^{=1})\,\mathbf{n}}} - (\mathbf{n}\cdot\nabla h)\{\underbrace{[\overbrace{(\mathbf{v}\cdot\nabla)\mathbf{n}}^{=\mathbf{v}}]\cdot\mathbf{n}}_{=0}\}\mathbf{n}\Big|_{\mathbf{n}} \\
&= \mathbf{0}
\end{aligned}
$$

where corresponding terms (that cancel each other) have been underlined. Consequently, if (26) is satisfied, then $\mathrm{d}\mathbf{x}_h$ maps the tangent space of S onto itself. Since the normal of S at $\mathbf{n}$ equals $\mathbf{n}$, the inverse of the differential is minus the Weingarten map. □

In particular, the principal directions of M are the eigenvectors of $\mathrm{Hess}_S h$ and if λ is an eigenvalue of $\mathrm{Hess}_S h$, then $-(\lambda + h)$ is a principal radius of curvature. Since the Weingarten map of the image of $\mathbf{x}_h$ is invertible at all points, none of the principal curvatures in any point can be zero. Thus, in the regular case, only curves without inflection points and hypersurfaces without parabolic points can be obtained from support functions.

The previously presented results are independent of a particular parameterization of S. In order to analyze and to visualize the surfaces for $d = 1, 2$, the following parameterizations may be useful.

Example 5 $(d = 1)$ Consider the parameterization

$$\mathbf{n} = \mathbf{n}(\theta) = (\sin\theta, \cos\theta)^\top, \quad \theta \in [-\pi, \pi) \tag{28}$$

of $S \subset \mathbb{R}^2$, which gives the outward normal. If $h = h(\theta)$ is a C^1 support function [3] then

$$\mathbf{x}_h = h(\theta)\mathbf{n}(\theta) + h'(\theta)\mathbf{n}'(\theta) \tag{29}$$

If $h = h(\theta)$ is C^2, then

$$\mathrm{d}\mathbf{x}_h : \mathbf{n}'(\theta) \mapsto (h''(\theta) + h(\theta))\mathbf{n}'(\theta). \tag{30}$$

In particular, it is easy to see that the curvature of the curve $\mathbf{x}_h(\theta)$ equals $\kappa(\theta) = -(h''(\theta) + h(\theta))^{-1}$.

[3] Here we simply write $h(\theta)$ instead of $h(\mathbf{n}(\theta))$.

Example 6 ($d = 2$) Consider the parameterization

$$\mathbf{n} = \mathbf{n}(\phi, \psi) = (\sin\phi \sin\psi, \cos\phi \sin\psi, \cos\psi), \quad \phi \in [-\pi, \pi)^\top, \ \psi \in [0, \pi) \tag{31}$$

of $S \subset \mathbb{R}^3$. If $h = h(\phi, \psi)$ is a C^1 support function, then

$$\mathbf{x}_h(\phi, \psi) = h(\phi, \psi)\mathbf{n} + \frac{h_\phi(\phi, \psi)}{\sin^2(\psi)}\mathbf{n}_\phi + h_\psi(\phi, \psi)\mathbf{n}_\psi, \tag{32}$$

where the subscripts indicate the partial derivatives. If $h(\phi, \psi)$ is C^2, then the differential $\mathrm{d}\mathbf{x}_h$ is defined by its values on the basis $\mathbf{n}_\phi, \mathbf{n}_\psi$ of $T_\mathbf{n}S$,

$$\begin{aligned} \mathrm{d}\mathbf{x}_h \Big|_\mathbf{n} (\mathbf{n}_\phi) &= (h + \frac{h_{\phi\phi}}{\sin^2\psi} + \frac{h_\psi \cos\psi}{\sin\psi})\mathbf{n}_\phi + (-\frac{h_\phi \cos\psi}{\sin\psi} + h_{\psi\phi})\mathbf{n}_\psi \\ \mathrm{d}\mathbf{x}_h \Big|_\mathbf{n} (\mathbf{n}_\psi) &= (\frac{h_{\phi\psi}}{\sin^2\psi} - \frac{h_\phi \cos\psi}{\sin^3\psi})\mathbf{n}_\phi + (h + h_{\psi\psi})\mathbf{n}_\psi. \end{aligned} \tag{33}$$

4.2 The linear space of quasi–convex shapes

It will be convenient to interpret $\mathbf{x} \in C^0(S, \mathbb{R}^{d+1})$ as an *oriented shape*, i.e. as a set $\mathrm{Im}(\mathbf{x}) \subset \mathbb{R}^{d+1}$ together with normal vectors $\mathbf{n}$ attached at $\mathbf{x}(\mathbf{n})$ for all $\mathbf{n}$. If the mapping $\mathbf{x}$ is not injective, then more than one normal can be attached to some points of the shape. Also, for a general function $\mathbf{x}$, the surface normal at $\mathbf{x}(\mathbf{n})$ may be different from $\mathbf{n}$. However, they are identical at regular points of $\mathbf{x} = \mathbf{x}_h$.

Definition 7 *The oriented shapes obtained as* $\mathrm{Im}(\mathbf{x}_h)$ *from* C^1 *support functions* $h \in C^1(S, \mathbb{R})$ *will be called oriented quasi-convex shapes (curves, surfaces). The space of all oriented quasi-convex shapes will be denoted* $\mathcal{Q}_d$.

$\mathcal{Q}_d$ possess the structure of a real linear space with respect to convolution (addition of the support functions) and homotheties with center 0 (multiplication by scalar).

Remark 8 The *convolution* of two oriented surfaces $\mathcal{A}$, $\mathcal{B}$ with associated unit normal fields $\mathbf{n} = \mathbf{n}(\mathbf{b})$, $\mathbf{m} = \mathbf{m}(\mathbf{b})$ for $\mathbf{a} \in \mathcal{A}$, $\mathbf{b} \in \mathcal{B}$ is the surface

$$\mathcal{A} \star \mathcal{B} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{n}(\mathbf{a}) = \mathbf{m}(\mathbf{b})\}. \tag{34}$$

This notion is closely related to Minkowski sums. In the general case, the boundary of the Minkowski sum of two sets A, B is contained in the convolution of the two boundary surfaces $\delta A, \delta B$. In the case of convex sets, the boundary of the Minkowski sum and the convolution surface are identical. If one of the surfaces is a sphere, then the convolution is a one–sided offset surface. See [19,21] for more information and related references.

Example 9 We analyze the oriented shapes which are associated with the simplest possible support functions.

Table 1
Geometric operations and corresponding changes of the support function.

Geometric operation	Modified support
Translation by vector $\mathbf{v}$	$h^{\mathbf{v}}(\mathbf{n}) = h(\mathbf{n}) + \mathbf{n} \cdot \mathbf{v}$
Rotation by matrix $\mu \in SO(d+1)$	$h^{\mu}(\mathbf{n}) = h(\mu^{-1}(\mathbf{n}))$
Scaling by factor $c \in \mathbb{R}$	$h^{c}(\mathbf{n}) = c\, h(\mathbf{n})$
Offsetting with distance d	$h^{d}(\mathbf{n}) = h(\mathbf{n}) + d$
Change of orientation (reversion of all normals)	$h^{-}(\mathbf{n}) = -h(-\mathbf{n})$

- If $h(\mathbf{n}) = c$ is a constant function, then $\mathbf{x}_h(\mathbf{n}) = c\mathbf{n}$. The corresponding shape is the sphere with the radius $|c|$ oriented by outer (if $c > 0$) or inner (if $c < 0$) normals.
- If $h(\mathbf{n}) = \mathbf{n} \cdot \mathbf{v}$, where $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^{d+1}$ is a constant vector, then

$$\nabla_S(h)\Big|_{\mathbf{n}} = \pi_{\mathbf{n}}(\mathbf{v}) = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}, \tag{35}$$

  hence $\mathbf{x}_h(\mathbf{n}) = \mathbf{v}$. The corresponding shape is the single point $\mathbf{v}$ with attached unit normals in all directions.

Proposition 10 *The set of quasi-convex shapes $\mathcal{Q}_d$ is closed under the geometric operations of translation, rotation, scaling, offsetting, convolution and change of orientation (reversion all normals).*

Proof. Table 1 summarizes how these geometrical operation affect the corresponding support function. □

In particular, the envelope operator commutes with any special orthogonal transformation $\mu \in SO(d+1)$,

$$\mu \circ \mathcal{E} = \mathcal{E} \circ \mu. \tag{36}$$

4.3 Smoothness

If h is C^k, then $\mathbf{x}_h = \mathcal{E}(h)$ is C^{k-1}. However if it defines a regular hypersurface $M = \mathbf{x}_h(U)$, then M is even C^k. More precisely, there exists a C^k parameterization of the hypersurface. We discuss this in the following proposition.

Proposition 11 *Let $h : U \to \mathbb{R}$, where U is an open subset of S, be a C^k function. Recall that $\pi_{\mathbf{n}}$ is the orthogonal projection from $\mathbb{R}^{d+1}$ to $T_{\mathbf{n}}U = T_{\mathbf{x}_h(\mathbf{n})}M$ and that $\pi_{\mathbf{n}} \Big| M$ is a local homeomorphism. If $k \geq 2$ and (26) holds at all points $\mathbf{n} \in U$, then the inverse projection $(\pi_{\mathbf{n}} \Big| M)^{-1}$ is a local C^k parameterization.*

Proof. The regularity condition immediately shows that $(\pi_{\mathbf{n}} \Big| M)^{-1}$ is a local C^{k-1} parameterization. As the normal $\mathbf{n}$ is a C^{k-1} vector valued function too, an inspection of the proof of [7, Theorem 10.1] reveals that $(\pi_{\mathbf{n}} \big| M)^{-1}$ is of class C^k. $\square$

In the case $k = 1$ the regularity condition (26) does not make sense and instead it is essentially necessary to assume the existence of a tangent plane to show that M is of class C^1. However, in applications, the support function h will often be given as a piecewise C^∞ function. In this situation, it is possible to derive a simpler condition.

Proposition 12 *Let $h : U \to \mathbb{R}$, where U is an open subset of S, be a C^1 function. We assume that h is C^2 for all $\mathbf{n} \in U_0 \subseteq U$, where $U \setminus U_0$ is a collection of finitely many smooth sub–manifolds of dimension $d-1$ intersecting transversally in S, and we let $\pi_{\mathbf{n}}$ be as in the previous proposition. If there exists an $\epsilon > 0$ such that the function $\det(\mathrm{Hess}_S(h) + hI)$ satisfies either*

$$\forall \mathbf{n} \in U_0 : \det(\mathrm{Hess}_S(h) + hI) \Big|_{\mathbf{n}} > \epsilon \quad \text{or} \quad \forall \mathbf{n} \in U_0 : \det(\mathrm{Hess}_S(h) + hI) \Big|_{\mathbf{n}} < -\epsilon,$$

then $(\pi_{\mathbf{n}}|_M)^{-1}$ is a local C^1 parameterization around $\mathbf{x}_h(\mathbf{n})$ for all $\mathbf{n} \in U$.

Proof. From the previous proposition $M = \mathbf{x}(U)$ is a collection of C^2 patches. If two of these patches meets along a common C^2 boundary, then they meet with matching tangent spaces, so either they do form a C^1 hypersurface or they meet in a cuspidal 'edge' (of dimension $d-1$). Let $\mathbf{x}(\mathbf{n})$ be a point on the common boundary and choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_d$ for the tangent space $T_{\mathbf{n}}S^n$ such that $\mathrm{d}_{\mathbf{n}}\mathbf{x}\mathbf{v}_1, \ldots, \mathrm{d}_{\mathbf{n}}\mathbf{x}\mathbf{v}_{d-1}$ is tangent to the common boundary. If the two values of $\mathrm{d}_{\mathbf{n}}\mathbf{x}\mathbf{v}_d$ is on the same side of $\mathrm{span}\{\mathrm{d}_{\mathbf{n}}\mathbf{x}\mathbf{v}_1, \ldots, \mathrm{d}_{\mathbf{n}}\mathbf{x}\mathbf{v}_{d-1}\}$ then the hypersurface is C^1 at $\mathbf{x}(\mathbf{n})$. Thus, the hypersurface is C^1 if the two orientations of the tangent space $T_{\mathbf{x}(\mathbf{n})}\mathbf{x}(U)$ agrees, and as the orientation is determined by the sign of $\det(\mathrm{Hess}_S(h) + h)$, the result follows. $\square$

4.4 Norms

Next we discuss the relation between various norms of $h \in C^1(U, \mathbb{R})$ and $\mathbf{x}_h = \mathcal{E}(h) \in C^0(U, \mathbb{R}^{d+1})$, where $U \subseteq S$.

Proposition 13 *The point–wise equation*

$$\forall \mathbf{n} \in U : \|\mathbf{x}_h(\mathbf{n})\|^2 = |h(\mathbf{n})|^2 + \|\nabla_S h \Big|_{\mathbf{n}}\|^2 \tag{37}$$

implies

$$\|\mathbf{x}_h\|_2^2 = \|h\|_2^2 + \|\nabla_S h\|_2^2, \tag{38}$$

where $\|\cdot\|_2$ is the L^2 norm in $C^1(U, \mathbb{R})$ and $C^0(U, \mathbb{R}^{d+1})$, respectively, and

$$\|\mathbf{x}_h\|_\infty^2 \le \|h\|_\infty^2 + \|\nabla_S h\|_\infty^2, \tag{39}$$

where $\|\cdot\|_\infty$ is the L^∞ norm in $C^1(U, \mathbb{R})$ and $C^0(U, \mathbb{R}^{d+1})$, respectively.

Proof. The first equation (37) follows from $\mathbf{n} \cdot \nabla_S h \Big|_{\mathbf{n}} = 0$. $\square$

Proposition 14 *If $h \in C^2(S, \mathbb{R})$ satisfies (26) for all $\mathbf{n} \in S$, then*

$$\|\mathbf{x}_h\|_\infty = \|h\|_\infty. \tag{40}$$

Proof. The maximum of $\|\mathbf{x}_h\|^2 = |h|^2 + \|\nabla_S h\|^2$ is attained at a point where the gradient vanishes. Since

$$\nabla_S(h^2 + \nabla_S h \cdot \nabla_S h) = 2h\nabla_S h + 2\mathrm{Hess}_S h \nabla_S h = 2(\mathrm{Hess}_S h + h)\nabla_S h$$

this occurs at a point where $\nabla_S h \Big|_{\mathbf{n}} = 0$. At this point, (37) becomes $\|\mathbf{x}_h(\mathbf{n})\| = |h(\mathbf{n})|$ which is bounded by $\|h\|_\infty$, hence $\|\mathbf{x}_h\|_\infty \leq \|h\|_\infty$. On the other hand, the point–wise equation (37) gives $\|h\|_\infty \leq \|\mathbf{x}_h\|_\infty$. $\square$

This result is closely related a classical bound on the Hausdorff distance of convex shapes. If h_1, h_2 are the support functions of two closed convex hypersurfaces $\mathcal{C}_1, \mathcal{C}_2$ with outward pointing normals, then

$$\mathrm{dist}_{\mathrm{Hausdorff}}(\mathcal{C}_1, \mathcal{C}_2) = \|h_1 - h_2\|_\infty, \tag{41}$$

see [11].

In the case of a constant support function h, the inequality (39) is an equality.

Corollary 15 *The norm of the envelope operator $\mathcal{E}$ is 1 when considering the L^2 (resp. L^∞) norm of the domain space and the corresponding Sobolev norm of the image space.*

Remark 16 The regularity condition (26) is indeed necessary for (40), as shown by the following example. Let $d = 1$ and consider the parameterization (28) of S. Then $h(\theta) = \cos(2\theta)$ defines a C^2 function on S. The envelope $\mathbf{x}_h$ can be evaluated using (29),

$$\mathbf{x}_h(\theta) = (-3\sin(\theta) + 2\sin^3(\theta), 3\cos(\theta) - 2\cos^3(\theta))^\top,$$

see the first picture of Fig. 2. We obtain $\|\mathbf{x}_h\|_\infty = |\mathbf{x}_h(\pi/4)| = 2$ and $\|h\|_\infty = 1$.

5 Polynomial support functions

In this section we study the shapes corresponding to support functions obtained by restricting polynomials defined on $\mathbb{R}^{d+1}$ to S.

Definition 17 *A quasi-convex shape with a support function which is a restriction of a polynomial of degree k on $\mathbb{R}^{d+1}$ to S will be called* ***quasi-convex shape of degree*** k.

The set of all quasi-convex shapes of degree k forms a linear subspace of $\mathcal{Q}_d$ closed under all geometric operation listed in the Table 1. In particular, this set is (for $k > 0$) independent of the choice of the coordinate system, as the space of support functions contains linear polynomials (which correspond to translations) and this space is also invariant under rotations, see Proposition 10 and Eq. (36).

Proposition 18 *Any quasi–convex shape of degree k admits a rational parameterization of degree $2k + 2$.*

Proof. If the support function h is a polynomial of degree k, then both $h\mathbf{n}$ and $\nabla_S h = \nabla h - (\nabla h \cdot \mathbf{n})\mathbf{n}$ are restrictions of polynomials of degree $k + 1$ to S. Consequently, $\mathbf{x}_h = \mathcal{E}(h)$ is the restriction of a polynomial of degree $k + 1$. By composing it with a quadratic rational parameterization of S (which can be obtained via stereographic projection) we obtain a rational parameterization of degree $2k + 2$. □

5.1 Curves ($d = 1$)

Even simple polynomial support functions on the circle correspond to rather complicated and non-symmetric shapes. On the other hand, using the parameterization (28) of the circle, any such function can be expressed as a trigonometric polynomial in θ. The basis functions $\cos(k\theta)$ and $\sin(k\theta)$ lead to simple quasi-convex oriented shapes.

Lemma 19 *The hypocycloid generated by rolling a circle of radius r within a circle of radius R has the support function*

$$\mathbf{h}(\theta) = (R - 2r) \cos\left(\frac{R}{R - 2r}\theta\right) \tag{42}$$

with respect to the parameterization (28).

Proof. We choose the coordinates such that the fixed circle is centered at the origin, while for $\theta = 0$ the center of the rolling circle is located at $(0, R-r)^\top$, and the tracing point at $(0, R-2r)^\top$, see Figure 1, grey circle. The associated normal is the "outer" normal $(0, 1)^\top$. Suppose that the small circle rotates through angle α arriving at the position represented by the small black circle. By the definition of the hypocycloid we have $\measuredangle LSP = \alpha$ and $\measuredangle YOT = (r/R)\,\alpha$. Moreover, the normal of the hypocycloid at the point P passes through the point T of contact of both circles. The tangent KP is therefore perpendicular to the segment TP and passes through L. Due to the similarity of triangles $\triangle KOL \sim \triangle PTL$,

$$\measuredangle KOT = \measuredangle LTP = \frac{\measuredangle LSP}{2} = \frac{\alpha}{2} \tag{43}$$

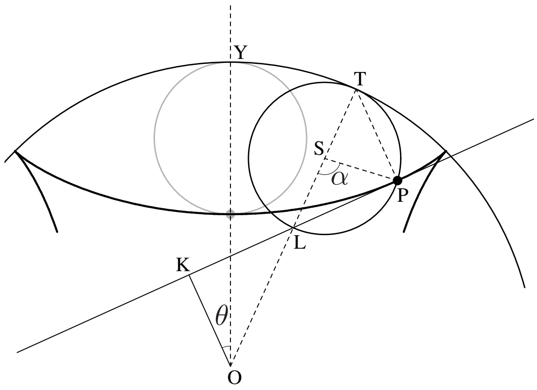


Fig. 1. A hypocycloid and its tangent.

and the angle θ of the normal KO at P equals

$$\theta = \measuredangle KOY = \measuredangle YOT - \measuredangle KOT = -\frac{R-2r}{2R}\alpha. \tag{44}$$

Finally we obtain the distance of the tangent KP from the origin

$$h(\theta) = |KO| = \frac{R-2r}{2r}|TP| = \frac{R-2r}{2r}2r\cos\frac{\alpha}{2}. \tag{45}$$

which implies (42). $\square$

By choosing $r = \frac{k-1}{2}$ and $R = k$ in (42) we obtain the support functions

$$\mathbf{h}(\theta) = \cos(k\theta), \quad k \in \mathbb{N}$$

defined over the entire circle S. The corresponding shapes are closed hypocycloids with ratio of circle radii $k : \frac{k-1}{2}$. They will be called *hypocycloid of degree* k, see Fig. 2. If k is odd, the hypocycloid is traced twice, but with opposite normals.

Proposition 20 *The hypocycloid of degree k is a quasi-convex curve of degree k. Any quasi-convex curve of degree k can be obtained as the convolution of a circle, a point and at most $k-1$ hypocycloids (suitably rotated and scaled). Only hypocycloids of degree less or equal to k occur, each at most once.*

Proof. Using (28), the support function $\cos(k\theta)$ on S can be expressed a polynomial of degree k in $\cos(\theta) = y$,

$$\cos(k\theta) = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^{\ell} \binom{k}{2\ell} y^{k-2\ell}(1-y^2)^{\ell}. \tag{46}$$

The hypocycloid of degree k is therefore a quasi-convex curve of degree k. For any quasi–convex curve of degree k, the support function has a finite Fourier expansion

$$p(x,y) = p_0 + \sum_{i=1}^{k}(c_i\cos(i\theta) + s_i\sin(i\theta)). \tag{47}$$

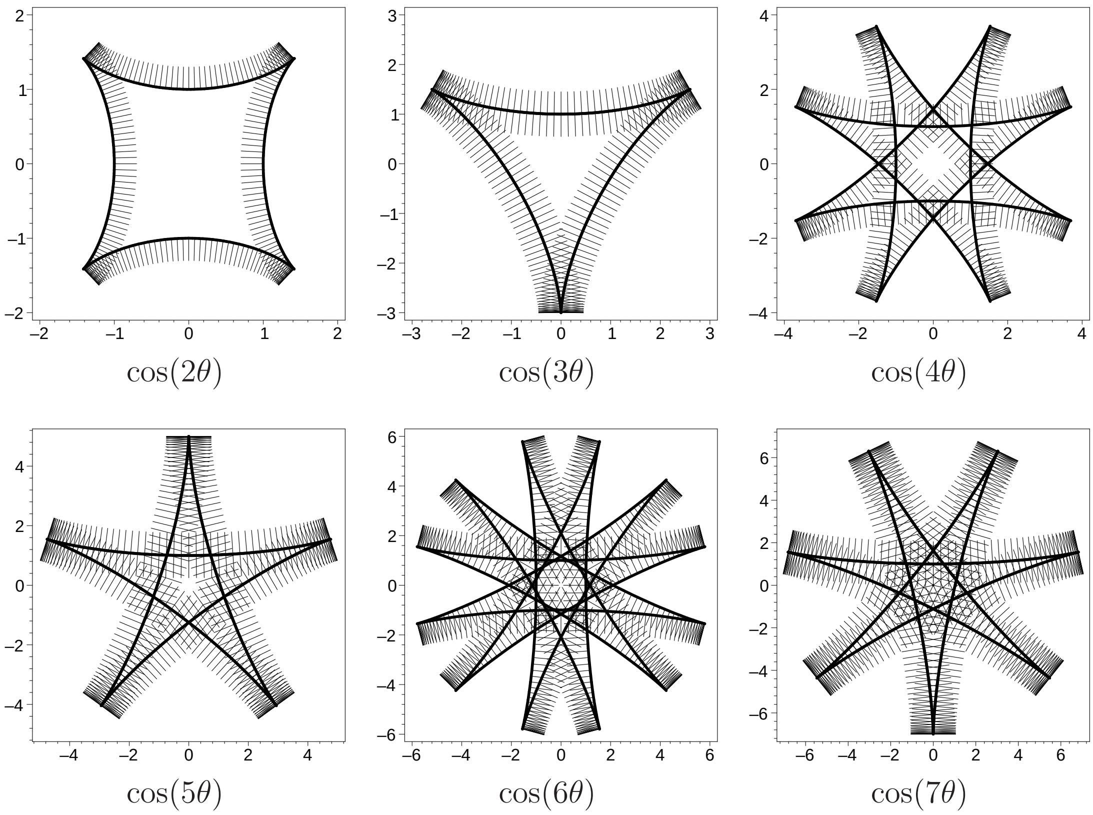


Fig. 2. Hypocycloids of degree 2 to 7 with attached normals. Note the different scaling.

For each i, we can find an angle θ_i such that $c_i = m_i \cos(i\theta_i)$ and $s_i = m_i \sin(i\theta_i)$, where $m_i = \sqrt{c_i^2 + s_i^2}$, hence

$$p(x, y) = p_0 + \sum_{i=1}^{k} m_i \cos(i(\theta - \theta_i)). \tag{48}$$

Consequently, the curve is obtained as the convolution of a oriented circle with radius p_0, the point $(m_1 \sin(\theta_1), m_1 \cos(\theta_1))^\top$ and of $k - 2$ rotated hypocycloids obtained for $i = 2, \ldots, k$. $\square$

Example 21 Consider the polynomial

$$p(x, y) = \tfrac{32}{5}\, y^4 + \tfrac{7}{3}\, x^3 + 4\, x^2 y - \tfrac{17}{4}\, xy^2 - \tfrac{7}{5}\, y^3 - \tfrac{3}{4}\, x^2 + 2\, xy - \tfrac{86}{15}\, y^2 + 3\, x - \tfrac{14}{5}\, y + \tfrac{154}{5}.$$

The corresponding quasi–convex curve is shown in Figure 3, left. By computing the Fourier coefficients, one finds that it is equal to

$$\begin{aligned} h(\theta) = \tfrac{719}{24} + \left[\tfrac{57}{20}\, \cos(\theta) + \tfrac{59}{16}\, \sin(\theta)\right] + \left[\tfrac{17}{24}\, \cos(2\,\theta) + \sin(2\,\theta)\right] + \\ + \left[-\tfrac{27}{20}\, \cos(3\,\theta) - \tfrac{79}{48}\, \sin(3\,\theta)\right] + \tfrac{4}{5}\, \cos(4\,\theta)\,. \end{aligned} \tag{49}$$

with respect to the parameterization (28). The original curve is therefore obtained as a convolution of the circle with radius $\frac{719}{24}$, of the point $[\frac{57}{20}, \frac{59}{16}]$ and of the three

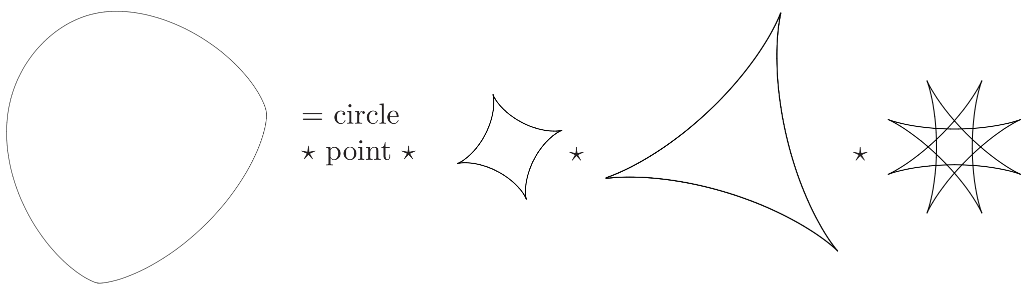


Fig. 3. A curve (left) and its three hypocycloidical components.

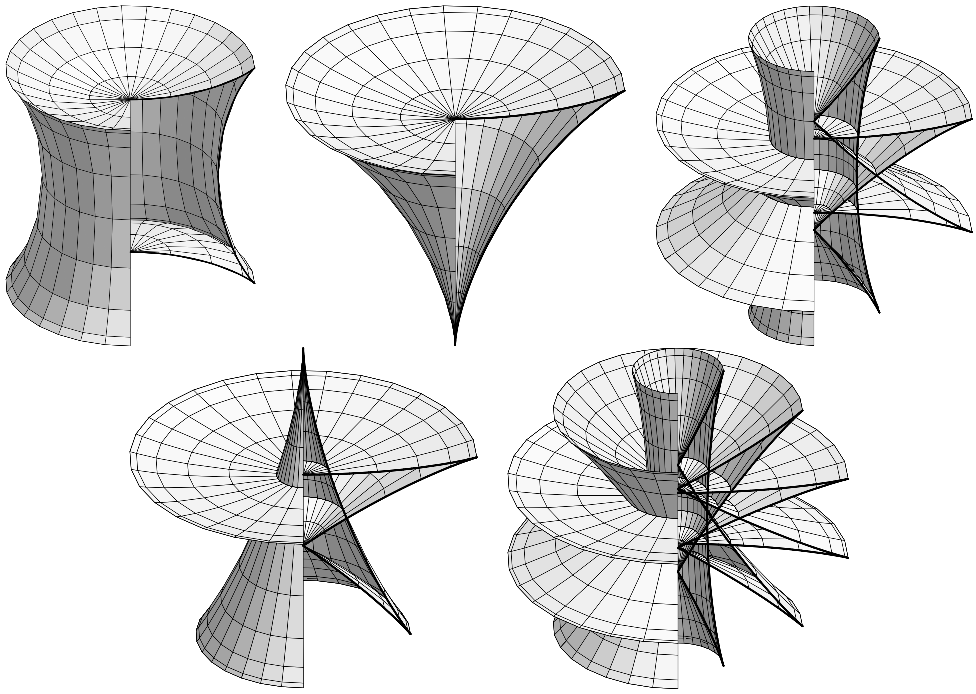


Fig. 4. HCR-surfaces of degrees 2 to 6.

hypocycloids shown in Fig. 3, right.

5.2 Hypersurfaces $(d \geq 2)$

In order to extend the previous results to an arbitrary dimension, we define what we call an *HCR-shape* of degree k. It is the shape with the support function defined as the restriction of

$$\sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^{\ell} \binom{k}{2\ell} x_1^{k-2\ell}(1-x_1^2)^{\ell} \tag{50}$$

considered as a polynomial in $x_1, x_2, ..., x_{d+1}$ (though only x_1 appears). In particular, if $d = 2$, then this HCR shape is simply the surface of **R**evolution obtained by rotating the **H**ypo**C**ycloid (42) of degree k around the x_2–axis. Similar interpretations exist for higher dimensions. See Fig. 4 for examples of HCR-surfaces (i.e. $d = 2$).

Proposition 22 *The HCR-shape of degree k is a quasi-convex shape of degree k. Moreover, any quasi-convex shape of degree k can be obtained as a convolution of a sphere, a point and at most*

$$\frac{2k+n}{k}\binom{n+k-1}{n} - (d+1)$$

HCR-shapes. Only HCR-shapes of degree $i \leq k$ occur, each at most

$$\frac{2k+n-1}{k+n-1}\binom{k+n-1}{n-1}$$

times.

Proof. The HCR-shape of degree k has the support function given by restriction of the polynomial (50) and therefore it is a quasi-convex shape of degree k. The support function for a HCR-shape is of the type (19), so the proposition is obtained by applying Lemma 1. □

6 Approximation of support functions

Based on the previous results, we show how to approximate any quasi–convex curve or surface by rational curves or surfaces with rational offsets.

6.1 Harmonic expansion

The support function of a given quasi–convex shape can be approximated by its harmonic ($d = 1$: Fourier) expansion up to certain degree. The corresponding shape approximates the original one with an accuracy which can be determined from the support function, due to Proposition 13. These approximations preserve all original symmetries.

This approach is particularly well suited for "smooth" shapes. Due to Proposition 18, we obtain approximations of the original shape by rational curves (and surfaces) with rational offsets.

We illustrate this observation by two examples.

Example 23 In order to demonstrate the approximation power of the Fourier expansion, we approximate the planar shape with the support function

$$h(\theta) = \sin(\sin(\theta)) + \cos(\cos(\theta)) + \frac{1}{2}$$

(see the grey curve in Fig. 5) by shapes of finite degree (black curves).

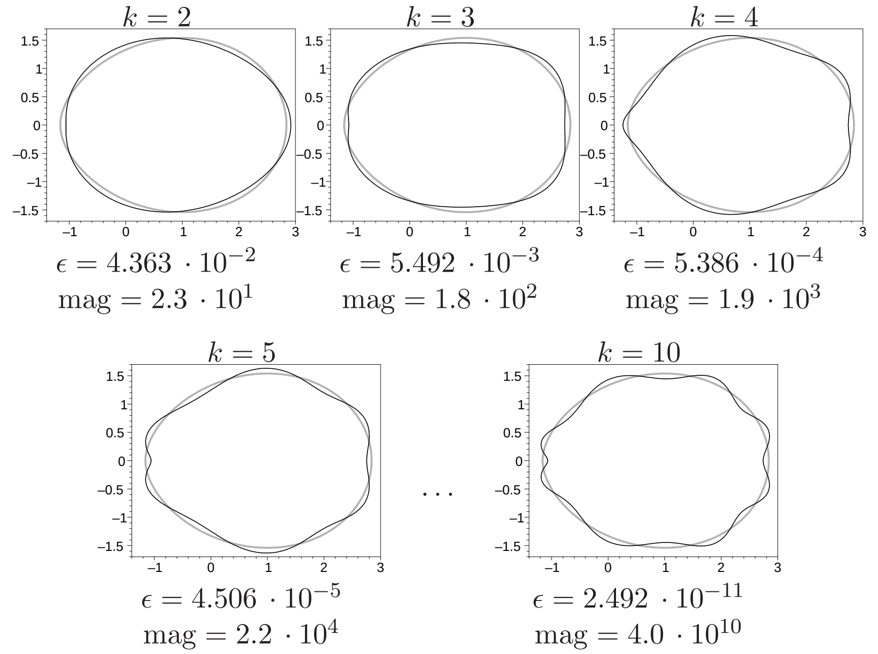


Fig. 5. Approximations of a given quasi–convex curve (grey) by (rational) shapes of finite degree k. The Hausdorff distance ϵ between the target and the approximation shapes (printed below each figure) is invisible for $k > 3$. In order to show the mutual position of the curves, we magnified the gap between them by coefficient mag.

Example 24 We approximate an ellipsoid with axes of lengths 1, $\sqrt{2}$ and 2. The support function is the restriction of $h_0 = \sqrt{x^2 + 2y^2 + 4z^2}$ to S. Figure 6 shows the approximation of the ellipsoid and of its offsets based on the harmonic expansion up to degree 6, which corresponds to a rational parametric representation of degree 14. The error of the shape approximation is 0.00187, or about 0.05% of the biggest diameter of the ellipsoid.

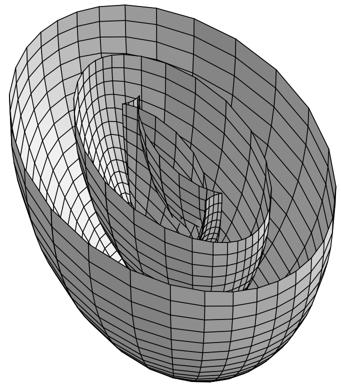


Fig. 6. Parametric rational approximation of degree 14 of the ellipsoid (outer shape) and of its two interior offsets at distances 0.45 and 0.9.

Table 2
Approximation error of a biquadratic tensor product patch.

Degree	Error	Degree	Error	Degree	Error	Degree	Error
2	$3.86\ 10^{-1}$	4	$2.80\ 10^{-2}$	6	$1.38\ 10^{-3}$	8	$6.32\ 10^{-5}$
3	$1.09\ 10^{-1}$	5	$6.50\ 10^{-3}$	7	$2.66\ 10^{-4}$	9	$1.37\ 10^{-5}$

6.2 Localized approximation

In many cases, only a surface patch may be given, and the use of a more local technique than global harmonic expansion may be more appropriate. We suppose that points $\mathbf{X}_i$ and associated unit normals $\mathbf{n}_i$ sampled from a surface patch are given. Consequently, $\mathbf{X}_i \cdot \mathbf{n}_i$ are the values and $\mathbf{X}_i - \mathbf{X}_i \cdot \mathbf{n}_i$ are the gradients of the support function of the patch at the point $\mathbf{n}_i$ of S.

In order to approximate the given surface by a surface with rational offsets, we are looking, within a given space $\mathcal{H}$, for the support function h approximating these values and gradients in the least-squares sense. More precisely, we solve the quadratic minimization problem

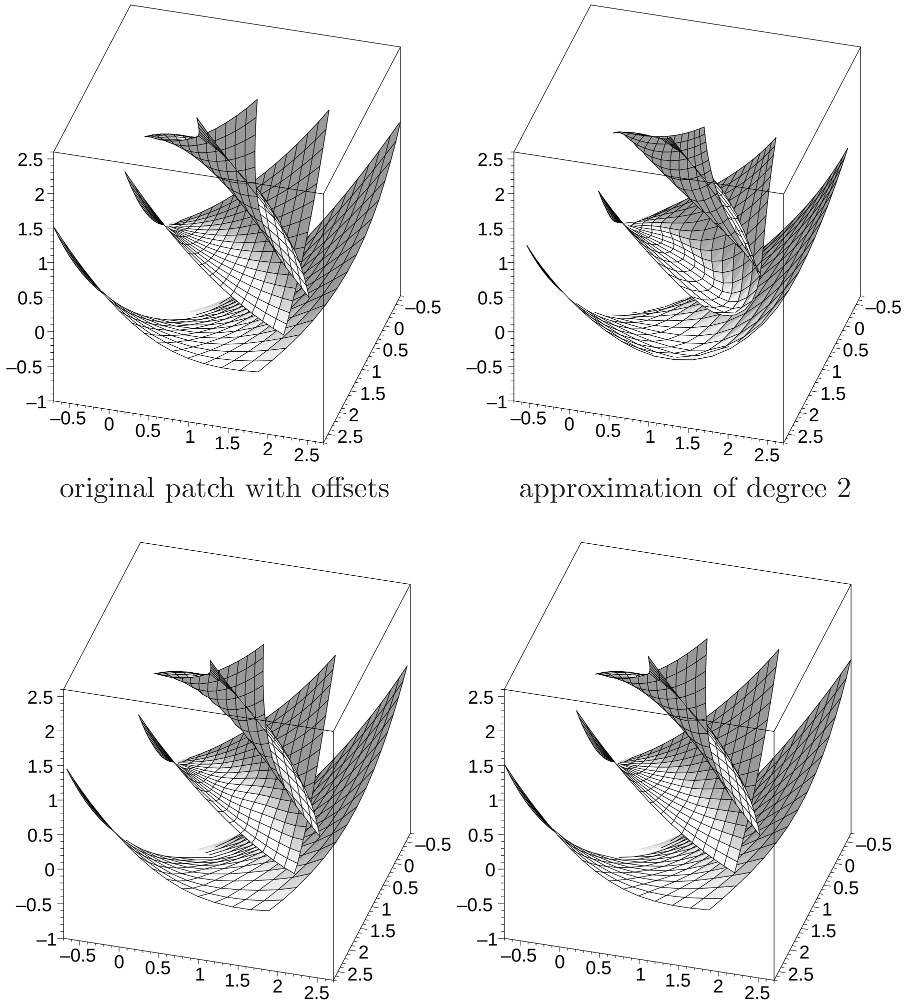
$$\min_{h \in \mathcal{H}} \left(\sum_{i=1}^{N} (h(\mathbf{n}_i) - \mathbf{X}_i \cdot \mathbf{n}_i)^2 + \sum_{i=1}^{N} \left\| \nabla_S h \,\Big|_{\mathbf{n}_i} - \mathbf{X}_i + \mathbf{X}_i \cdot \mathbf{n}_i \right\|^2 \right), \tag{51}$$

where $\mathcal{H}$ is a suitable linear space of support functions [4]. The unique minimum can be computed by solving a linear system of equations where unknowns are coefficients of h with respect to some basis of $\mathcal{H}$.

In our example (see below) we considered $\mathcal{H}$ to be (restrictions of) polynomials up to degree k. As a basis of this space one may choose the monomials of total degree k and $k-1$, i.e. the basis $\{x^p y^q z^r : (k-1) \leq p+q+r \leq k\}$. Clearly, it is also possible to use other spaces of functions, such as piecewise polynomials (i.e., spherical spline functions, see [1]).

Example 25 We consider a biquadratic polynomial tensor-product patch, see Figure 7. We sample N points $[\mathbf{X}_i]_{i=1}^N$ and we compute the unit normals $[\mathbf{n}_i]_{i=1}^N$ at these points. In our example we considered $N = 256$ points sampled at a regular grid in the parameter domain. As a result we obtain an approximation of the original patch by a piece of quasi-convex surface of degree k. Simultaneously we obtain approximations of all offsets within the same error. Table 2 and Figure 7 show and visualize the approximation error and its improvement for increasing degree of the support function.

[4] The summation in (51) can be seen as simple numerical integration, and the objective function could be defined using an integral.



original patch with offsets — approximation of degree 2

approximation of degree 3 — approximation of degree 4

Fig. 7. Approximations of the biquadratic patch and its offsets.

6.3 Piecewise linear approximation

As the simplest instance of spherical splines, we consider piecewise linear support functions which are defined on a triangulation of the Gaussian sphere. The segments of this function are restrictions of linear polynomials of the form $ax + by + cz$ to the sphere. They can be pieced together along great circular arcs, so as to form a globally continuous function. This simple class of spline functions can be used to interpolate the values of the support function at the vertices of the underlying spherical triangulation.

The associated surface cannot be obtained directly from the envelope operator, since the support function is not differentiable. Still one may associate a piecewise linear surface with it, which is the envelope of the family of planes (22). Its facets and vertices correspond to the vertices and triangles of the piecewise linear function on the Gaussian sphere, respectively. See [2] for a more detailed discussion, which also addresses the problem of regularity of the facets.

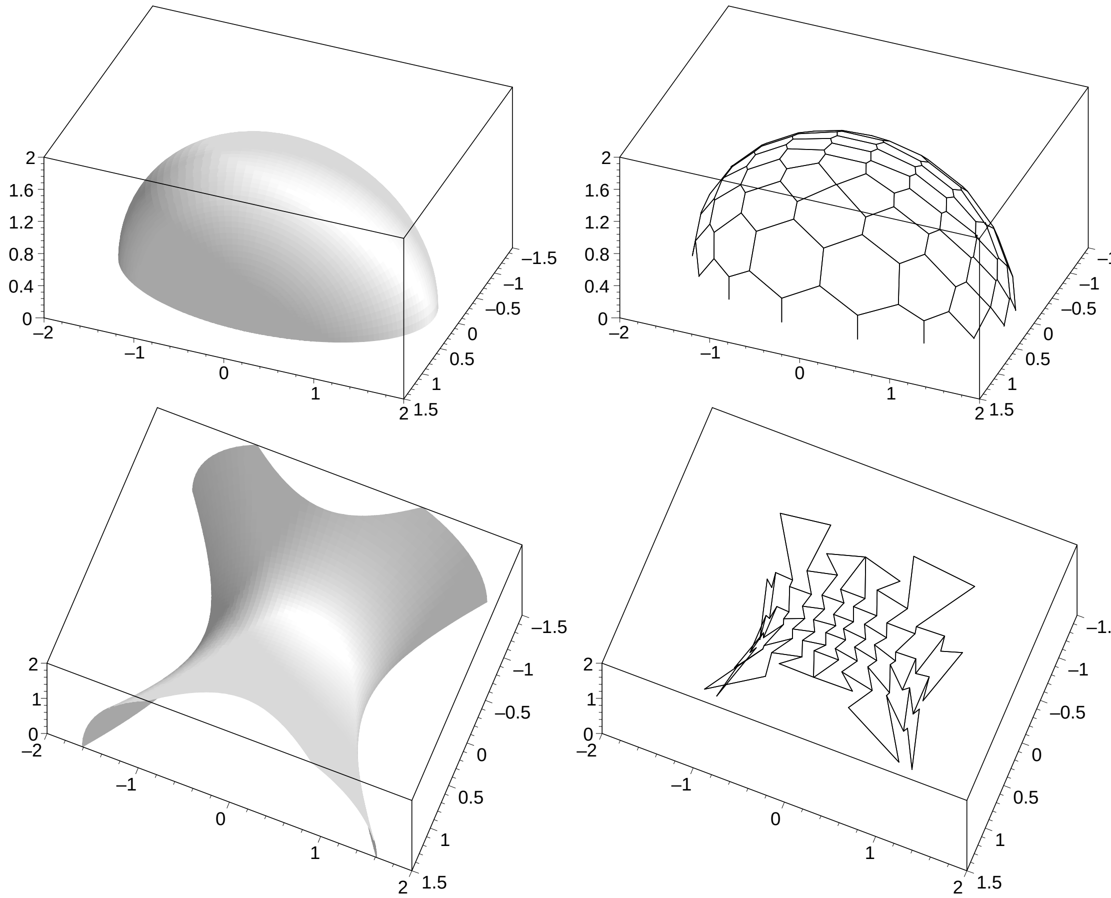


Fig. 8. Support function based approximations of elliptic and hyperbolic surfaces with piecewise linear surfaces.

Example 26 We consider the support functions which have been obtained by piecewise linear interpolation of the support functions of two quadric surfaces, see Fig. 8. Consequently, each facet of the piecewise linear surface is the tangent plane of the original surface at the point with the same normal. In the case of an ellipsoid, which contains only elliptic points, we obtain a mesh which consists mostly if convex hexagons (see Figure 8, top row). In the case of a hyperboloid of one sheet, which contains only hyperbolic points, we get a mesh which consists mostly of bow–tie–shaped non–convex hexagons (see Figure 8, bottom row).

7 Conclusion

In this paper we explored several aspects of the representation of curves and surfaces by (piecewise) polynomial support functions. The corresponding shapes are very well suited to define a set of curves and surfaces which is closed under convolutions and offsetting. Similar results can be obtained for other linear spaces of support functions (which should – of course – contain linear polynomials).

As a matter of future research, we aim at extending these results to curves and surfaces with inflections resp. with parabolic points. For instance, one might consider support–like functions defined on other surfaces than spheres. As another interesting questions one may try to identify conditions on the Bézier control points and weights of rational curves and surfaces which guarantee that the curve / surface belongs to a class of quasi–convex shapes of a certain degree. Finally, it should be interesting to investigate curves and surfaces defined by rational support functions.

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