Parameterizing surfaces with certain special support functions, including offsets of quadrics and rationally supported surfaces

Martin Aigner and Bert Jüttler

Johannes Kepler University Linz, Austria

Laureano Gonzalez-Vega

Universidad de Cantabria, Santander, Spain

Josef Schicho

Radon Institute of Computational and Applied Mathematics, Linz, Austria

Abstract

We discuss rational parameterizations of surfaces whose support functions are rational functions of the coordinates specifying the normal vector and of a given non-degenerate quadratic form. The class of these surfaces is closed under offsetting. It comprises surfaces with rational support functions and non-developable quadric surfaces, and it is a subset of the class of rational surfaces with rational offset surfaces. We show that a particular parameterization algorithm for del Pezzo surfaces can be used to construct rational parameterizations of these surfaces. If the quadratic form is diagonalized and has rational coefficients, then the resulting parameterizations are almost always described by rational functions with rational coefficients.

Key words: Parameterization of surfaces, support function, offset surface, del Pezzo surface

1. Introduction

The support function representation of a surface is one of the classical tools in the field of convex geometry (see Bonnesen and Fenchel, 1987; Groemer, 1996; Gruber and Wills, 1993). It describes the surface as the envelope of its tangent planes, where the distance between the tangent plane and the origin is specified by a function of the unit normal vector. This representation is particularly well suited for discussing offsets surfaces, since the offsetting operation corresponds simply to the addition of constants. See Gravesen

Email addresses: {martin.aigner|bert.juettler}@jku.at, www.ag.jku.at (Martin Aigner and Bert Jüttler), gvega@matesco.unican.es, www.matesco.unican.es (Laureano Gonzalez-Vega), josef.schicho@oeaw.ac.at, www.ricam.oeaw.ac.at (Josef Schicho).

et al. (2008); Šír et al. (2008) for a detailed discussion of surfaces with polynomial and rational support functions.

The analysis and parameterization of offset curves and surfaces via techniques from symbolic computation and algebraic geometry has been the central topic of several publications. Alcazar et al. (2007) apply a method for computing critical sets of algebraic surfaces to the offsetting problem. Alcazar and Sendra (2007) study the local shape of offsets to algebraic curves. Landsmann et al. (2001) present an algorithm for parameterizing canal surfaces by decomposing a polynomial into a sum of squares. Canal surfaces can be seen as (generalized) offsets of space curves. Arrondo et al. (1997) give a theoretical analysis of the rationality and unirationality of offsets to hypersurfaces. Of course it is also possible to apply results about the parameterization of general rational surfaces (Schicho, 1998) to the case of offset surfaces. However, this is generally not practical because the implicit equation of the offset either is not known or it is too large to be treated with Schicho's algorithm.

Due to their importance in applications in Computer Aided Design, the case of offsets to quadric surfaces has attracted special attention. Moreover, these surfaces are (after planes) the simplest instance of offsets to a class of algebraic surfaces. With the help of techniques from Laguerre Geometry, Peternell and Pottmann (1998) derive a rational parameterization of the offsets of quadric surfaces. First, the quadric surface and its offsets are represented as the envelope of a one-parameter-family of quadratic cones of revolution. Then a parameterization of the envelope is found by a geometric algorithm, which involves the decomposition of a polynomial into a sum of squares, similar to the case of canal surfaces, see Landsmann et al. (2001). This decomposition requires a suitable field extension, which may make the use of exact symbolic computation techniques more difficult. Sendra and Sendra (2000) discuss generalized offsets of irreducible quadrics and show how to obtain rational parameterizations (if available) from the parameterization of the original quadric.

In this paper we apply the support function representation to a class of surfaces which generalizes both surfaces with rational support functions (see Gravesen et al., 2008) and offsets of quadric surfaces. More precisely, the support function is a rational function of the coordinates of the normal vector and of the square root of a single quadratic form. We show how to generate rational parameterizations of surfaces from this class. Consequently, this class is a subset of the class of rational surfaces with rational offset surfaces (see Pottmann, 1995).

Except for certain special cases, such as the offsets to two-sheeted hyperboloids of revolution, the presented parameterization algorithm does not require field extensions. Consequently, it produces parameterizations with rational coefficients, provided that the input surfaces have also been specified by support functions with rational coefficients.

The remainder of this paper is structured in five parts. The next section recalls the concept of the dual representation of non-developable algebraic hypersurfaces and analyzes its relation to support functions. Section 3 introduces the class of surfaces which is studied in this paper, and Section 4 discusses its parameterization via the envelope operator. The fifth section presents an algorithm for parameterizing the intersection of two special hyperquadrics in four-dimensional space and applies it to the parameterization problem. Finally we conclude the paper.

2. Dual representation of algebraic surfaces and support functions

We consider algebraic surfaces in the three–dimensional Euclidean space, which is identified with $\mathbb{R}^3$. Sometimes it will be helpful to use the projective closure $\bar{\mathbb{R}}^3$ of this space. Recall that a non-developable algebraic surface $\mathcal{S}$ in $\mathbb{R}^3$ has a *dual representation* of the form

$$F(h, \mathbf{n}) = 0 \tag{1}$$

where F is a homogeneous polynomial in h and $\mathbf{n} = (n_1, n_2, n_3)^\top$. The degree of F is called the *class* of the surface $\mathcal{S}$. The set of all planes

$$T_{h,\mathbf{n}} = \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{n}^\top \mathbf{x} = h\}, \quad F(h, \mathbf{n}) = 0, \tag{2}$$

forms the system of the *tangent planes* of the surface $\mathcal{S}$. The vector $\mathbf{n}$ is the normal vector. If $\mathbf{n}^\top \mathbf{n} = 1$, then the value of h is the oriented distance of the tangent plane to the origin.

If the partial derivative $\partial F/\partial h$ does not vanish at $(h_0, \mathbf{n}_0) \in \mathbb{R}^4$ and $F(h_0, \mathbf{n}_0) = 0$ holds, then (1) implicitly defines a function

$$\mathbf{n} \mapsto h(\mathbf{n}), \tag{3}$$

which is well-defined in a certain neighborhood of $(h_0, \mathbf{n}_0) \in \mathbb{R}^4$. The restriction of this function to the unit sphere

$$\mathbb{S} = \{\mathbf{n} \in \mathbb{R}^3 : \ \mathbf{n}^\top \mathbf{n} = 1\} \tag{4}$$

is then called the *support function* of the surface $\mathcal{S}$.

Remark 1. Note that the dual representation (1) does not require the normal vectors $\mathbf{n}$ to be normalized. However, the support function is only defined for unit normals. Whenever we use the support function, then its argument $\mathbf{n}$ will be assumed to be a unit vector.

Alternatively, we may consider

$$h : n_1 : n_2 : n_3 = 1 : x_1 : x_2 : x_3 \tag{5}$$

as homogeneous coordinates in $\mathbb{R}^3$. Then Eq. (1) defines the *dual surface* $\mathcal{D}$ associated with $\mathcal{S}$. The dual surface has the equation

$$F(1, \mathbf{x}) = 0. \tag{6}$$

The points (resp. tangent planes) of this surface $\mathcal{D}$ are obtained by applying the polarity with respect to the imaginary unit sphere to the tangent planes (resp. points) of the surface $\mathcal{S}$. This polarity identifies the homogeneous coordinates of points and of planes according to (5).

If a *parametric representation* of a surface is known, then the support function can be obtained as shown in the following example.

Example 2. We consider the algebraic surface of order 4 which possesses the quadratic parameterization $\mathbf{p}(u, v) = (u + v, u^2, v^2)$. The dual representation

$$F(h, \mathbf{n}) = n_1^2(n_2 + n_3) + 4hn_2n_3 = 0 \tag{7}$$

can be found by eliminating u and v from the three equations

$$\mathbf{n}^\top \frac{\partial}{\partial u}\mathbf{p} = 0, \quad \mathbf{n}^\top \frac{\partial}{\partial v}\mathbf{p} = 0, \quad \mathbf{n}^\top \mathbf{p} - h = 0. \tag{8}$$

Consequently, as F is a cubic homogeneous polynomial, this surface has class three. The dual surface $\mathcal{D}$ is a cubic monoid (see Johansen et al., 2008) with a unique singular point at the origin. The support function of the surface is the function

$$h(\mathbf{n}) = -\frac{n_1^2(n_2 + n_3)}{4n_2n_3}. \tag{9}$$

In this case, we have obtained even a unique rational support function. This was possible, as the given parameterization describes a non–developable quadratic polynomial surface (see Gravesen et al., 2008).

On the other hand, the support function can also be obtained directly from the *implicit equation* of a surface, as shown in the next example.

Example 3. The quadric surface with the equation

$$f(\mathbf{x}) = x_1^2 + \frac{1}{b}x_2^2 + \frac{1}{c}x_3^2 - 1 = 0, \tag{10}$$

where we assume that $b, c \neq 0$, has axis-aligned principal diameters with radii 1, $\sqrt{b}$ and $\sqrt{c}$. The dual representation

$$F(h, \mathbf{n}) = n_1^2 + bn_2^2 + cn_3^2 - h^2 \tag{11}$$

can be found by eliminating the four variables λ and $\mathbf{x} = (x_1, x_2, x_3)$ from the five $(= 3 + 1 + 1)$ equations

$$\mathbf{n} - \lambda \nabla f = 0, \quad \mathbf{n}^\top \mathbf{x} - h = 0, \quad f(\mathbf{x}) = 0. \tag{12}$$

The support functions of the surface take the form

$$h(\mathbf{n}) = \pm\sqrt{n_1^2 + bn_2^2 + cn_3^2}. \tag{13}$$

Remark 4. In the case of algebraic surfaces of higher degree, the elimination of λ and $\mathbf{x} = (x_1, x_2, x_3)$ from the equations (12) produces the dual equation of the surface. The support function is then implicitly defined by it, as described in the beginning of this section.

Finally we note that certain geometric operations correspond to simple modifications of the support functions:

(1) *Rotations* can be *composed* with the support function; the support function of $\varrho(\mathcal{S})$ is $h \circ \varrho$, where ϱ is a rotation around the origin.
(2) A *translation* by a vector $\vec{\mathbf{v}}$ correspond to the *addition* of the homogeneous *linear polynomial* $\vec{\mathbf{v}}^\top \mathbf{n}$ to the support function.
(3) The one-sided *offset* of a surface at distance δ can be obtained by adding the *constant* δ to the support function.
(4) The *reciprocal* support function $(1/h)$ describes the surface which is obtained by applying the polarity with respect to the unit sphere to the *pedal surface*.

In the remainder of the paper we consider a class of surfaces with a specific form of the support function.

3. A special class of support functions

We consider support functions of the form

$$h(\mathbf{n}) = R(Q, \mathbf{n}) \tag{14}$$

where $Q = \sqrt{\mathbf{n}^\top \mathbf{D} \mathbf{n}}$, $\mathbf{D} = \text{diag}(1,\text{b},\text{c})$ with $b, c \neq 0$, and R is a rational function of its four arguments Q and $\mathbf{n} = (n_1, n_2, n_3)^\top$. We can rewrite this function as

$$h(\mathbf{n}) = \frac{p_1(Q, \mathbf{n}) + p_2(Q, \mathbf{n})}{q(Q, \mathbf{n})}, \tag{15}$$

where the two functions p_2 and q are homogeneous polynomials in Q and $\mathbf{n} = (n_1, n_2, n_3)^\top$ of the even degree $2d$, and p_1 is a homogeneous polynomial of the odd degree $2d + 1$, where d is a non-negative integer. This can be proved by exploiting the observation that the terms can be multiplied by multiples of $\mathbf{n}^\top \mathbf{n}$, since this expression equals 1 on the unit sphere $\mathbb{S}$, similar to the proof of Lemma 2 in Gravesen et al. (2008).

Clearly, the class of surfaces with support functions of the form (14) comprises non-developable quadric surfaces and their offsets, see Example 3. It is closed under offsetting and translations.

Remark 5. More generally, one might consider square roots of a general quadratic form. In order to simplify the notation, we assume that it has been diagonalized and scaled such that the first coefficient is equal to 1. Consequently, we assume that an appropriate coordinate system has been chosen.

The corresponding dual equation (1) can be found by eliminating N and Q from the three equations

$$p_1(Q,\mathbf{n}) + p_2(Q,\mathbf{n})N - h\,q(Q,\mathbf{n}) = 0, \quad N^2 - \mathbf{n}^\top\mathbf{n} = 0, \quad Q^2 - \mathbf{n}^\top\mathbf{D}\mathbf{n} = 0. \tag{16}$$

The left-hand sides of all equations are homogeneous polynomials in h, N, Q and $\mathbf{n} = (n_1, n_2, n_3)^\top$. Consequently, the elimination of N and Q produces a homogeneous polynomial F. Note that the dual equation (1) then corresponds to the four support functions

$$h_{\epsilon_1,\epsilon_2}(\mathbf{n}) = \frac{p_1(\epsilon_1\sqrt{\mathbf{n}^\top\mathbf{D}\mathbf{n}},\mathbf{n}) + \epsilon_2 p_2(\epsilon_1\sqrt{\mathbf{n}^\top\mathbf{D}\mathbf{n}},\mathbf{n})}{q(\epsilon_1\sqrt{\mathbf{n}^\top\mathbf{D}\mathbf{n}},\mathbf{n})}, \quad \epsilon_1,\epsilon_2 \in \{\pm 1\}, \tag{17}$$

due to the sign ambiguities in N and D. This gives two pairs of support functions describing the same surface. Indeed, the support functions $h(\mathbf{n})$ and $h^*(\mathbf{n}) = -h(-\mathbf{n})$ describe the same surface, but with opposite orientations of the normals.

Remark 6. The three equations (16) define three hypersurfaces in the five dimensional space with the homogeneous coordinates $h : N : Q : n_1 : n_2 : n_3$. We briefly describe these surfaces and their relation to the dual surface $\mathcal{D}$ associated with the support functions (17):

- The first equation describes a hypersurface of degree $2d + 1$, where each point of the line
$$n_1 = n_2 = n_3 = Q = 0 \tag{18}$$
has multiplicity $2d$. It is therefore a very special instance of a monoid hypersurface, see Johansen et al. (2008). We call this surface an *axial monoid* with axis (18) .
- The remaining two surfaces describe two quadratic hypercones with two–dimensional generators and one-dimensional singular loci.
- The first three unit points of the projective coordinate system span three lines. One of them is the axis of the axial monoid, while the other two lines are the singular loci of the hypercones.
- The three hypersurfaces intersect in a two dimensional surface. The dual surface is obtained as its image by a central projection with the center line spanned by the two points $(0 : 1 : 0 : 0 : 0 : 0)$ and $(0 : 0 : 1 : 0 : 0 : 0)$ into the 3-plane $N = Q = 0$.

Example 7. We consider the support function

$$h(\mathbf{n}) = n_1\sqrt{n_1^2 + n_2^2 + 2n_3^2}. \tag{19}$$

In this case, we have $d = 1$, $\mathbf{D} = \mathrm{diag}(1, 1, 2)$ and

$$p_1(Q,\mathbf{n}) = 0, \quad p_2(Q,\mathbf{n}) = n_1 Q, \quad q(Q,\mathbf{n}) = n_1^2 + n_2^2 + n_3^2. \tag{20}$$

After eliminating Q and N from the equations (16) we arrive at the dual representation of the surface $\mathcal{S}$,

$$F(h,\mathbf{n}) = (n_1^2 + n_2^2 + n_3^2)h^2 - n_1^2(n_1^2 + n_2^2 + 2n_3^2). \tag{21}$$

Example 3 (continued). One of the support functions is $h = Q$ with $\mathbf{D} = \mathrm{diag}(1, b, c)$, hence $d = 0$, $p_1 = Q$, $p_2 = 0$, $q = 1$. In this special case, the first surface degenerates into the hyperplane $Q - h = 0$. If $p_2 = r \neq 0$ was a non-zero constant, then the support function $h' = Q + r$ would correspond to the offsets of the quadric, and the first surface would be the hyperplane $Q + rN - h' = 0$. In both cases, the dual surface is obtained by projecting the intersection of the two remaining quadrics with the hyperplane into three–dimensional space.

4. Parameterization using the envelope operator

First we introduce an operator that assigns to each support function a parameterization of the corresponding surface, where the parameter domain is the sphere or a subset thereof, cf. Šír et al. (2008).

Definition 8. Let $U \subset \mathbb{S}$ be an open subset of the unit sphere[1] and $h \in C^\infty(U, \mathbb{R})$ be a support function. We define the **envelope operator**

$$\mathcal{E} :\ C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^3) \tag{22}$$

which is defined via

$$\mathcal{E}(h) :\ U \to \mathbb{R}^3 :\ \mathbf{n} \mapsto h(\mathbf{n})\mathbf{n} + (\nabla_S h)(\mathbf{n}) \tag{23}$$

with the intrinsic gradient

$$(\nabla_S h)(\mathbf{n}) = (\nabla h)(\mathbf{n}) - \left(\mathbf{n}^\top [(\nabla h)(\mathbf{n})]\right) \mathbf{n}. \tag{24}$$

Remark 9. The intrinsic gradient (24) is the projection of the gradient in $\mathbb{R}^3$ onto the unit sphere, where we assume that h has been extended to the embedding space. Eq. (23) gives the envelope of the two-parameter family of planes $T_{h(\mathbf{n}),\mathbf{n}}$, see (2).

For any parameterization $\nu : \Omega \to U$ of $U \subseteq \mathbb{S}$ with the domain $\Omega \subseteq \mathbb{R}^2$, the mapping $\mathcal{E}(h) \circ \nu : \Omega \to \mathbb{R}^3$ is a parameterization of the corresponding open subset of the surface $\mathcal{S}$ in three–dimensional space. Clearly, if we apply the envelope operator $\mathcal{E}$ to a rational support function h and compose the result with a rational parameterization ν of the sphere, then we obtain a rational parameterization $\mathcal{E}(h) \circ \nu$ of the corresponding surface $\mathcal{S}$.

In the case of surfaces with support functions of the form (14) we have the following result.

Lemma 10. *If the five bivariate polynomials $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}[u, v]$ satisfy the two identities*

$$x_1^2 + x_2^2 + x_3^2 = x_4^2 \quad \text{and} \quad x_1^2 + b\,x_2^2 + c\,x_3^2 = x_5^2, \tag{25}$$

such that $(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4})$ is a rational parameterization of the unit sphere, then the mapping

$$(u, v) \mapsto \mathcal{E}(h)\left(\frac{x_1(u,v)}{x_4(u,v)}, \frac{x_2(u,v)}{x_4(u,v)}, \frac{x_3(u,v)}{x_4(u,v)}\right) \tag{26}$$

is a piecewise rational parameterization of the surface which is defined by the support function $h(\mathbf{n})$ of the form (14).

Proof. If the support function has the form (14), then $\mathcal{E}(h)$ as defined in (23) contains only rational functions of $\mathbf{n}$ and $\sqrt{\mathbf{n}^\top \mathbf{D}\mathbf{n}}$. The rational parameterization of the unit sphere $\nu = (\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4})$ can be composed with the envelope operator $\mathcal{E}(h)$. After replacing the square root $\sqrt{\mathbf{n}^\top \mathbf{D}\mathbf{n}}$ in (26) with $|x_5|$ one obtains a piecewise rational parameterization of the surface. □

[1] U is the intersection of an open set with respect to the Euclidean topology in $\mathbb{R}^3$ with the unit sphere $\mathbb{S}$.

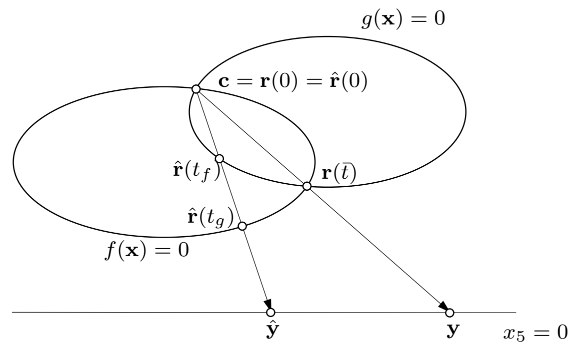


Figure 1. Stereographic projection of two intersecting quadrics

The next section discusses how to generate quintuples of bivariate polynomials that satisfy the assumptions of the Lemma.

5. Intersections of special hyperquadrics in four-dimensional space

The two identities (25) define two quadric surfaces

$$\begin{aligned} f(x_1, x_2, x_3, x_4, x_5) &= x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0 \\ g(x_1, x_2, x_3, x_4, x_5) &= x_1^2 + b\,x_2^2 + c\,x_3^2 - x_5^2 = 0 \end{aligned} \tag{27}$$

in four–dimensional real projective space with homogeneous coordinates $x_1 : x_2 : x_3 : x_4 : x_5$. The intersection is a two-dimensional del Pezzo surface (see Schicho, 2005).

We assume that the input satisfies $b \neq c$. See Remark 14 for a discussion of the special case $b = c$. We parameterize the intersection by applying the following algorithm.

Algorithm 11. Parameterization of the intersection of (27), where $b \neq c$.

(1) Find a point $\mathbf{c}$ on the intersection of the two quadrics.
(2) Apply stereographic projection with center $\mathbf{c}$ into a three-dimensional subspace to the intersection surface. This gives a cubic surface k.
(3) Find a straight line $\mathbf{l}$ on the cubic surface k and parameterize it linearly with parameter u.
(4) For each point $\mathbf{l}(u)$ on the line, compute the tangent plane $\mathbf{q}(u)$ of the cubic k.
(5) The intersection of the tangent plane $\mathbf{q}(u)$ with the cubic surface k gives a conic section, which is parameterized with the parameter v.
(6) Lift the parameterization of k back into the five-dimensional space.

Now we describe the six steps of the algorithm in more detail.

Step (1) We simply observe that the point $\mathbf{c} = (1, 0, 0, 1, 1)^\top$ lies on both quadrics, hence it is also contained in the intersection.

Step (2) We apply stereographic projection with center $\mathbf{c}$ and project the intersection of both quadrics into the plane $x_5 = 0$. More precisely, for each point $\mathbf{y} = (y_1, y_2, y_3, y_4, 0)$ we consider the line

$$\mathbf{r}(t) = (1 - t)\mathbf{c} + t\mathbf{y}, \tag{28}$$

see Figure 1 for a two-dimensional sketch. A point $\mathbf{y}$ belongs to the image of the intersection if and only if the line (28) intersects both quadrics in the same point. Equivalently, there exists a parameter $\bar{t} \neq 0$ such that the equations

$$f((1 - \bar{t})\mathbf{c} + \bar{t}\mathbf{y}) = 0, \quad g((1 - \bar{t})\mathbf{c} + \bar{t}\mathbf{y}) = 0 \tag{29}$$

are simultaneously satisfied. This can characterized by the resultant

$$k(\mathbf{y}) = \mathrm{Res}(\frac{1}{t} f((1-t)\mathbf{c} + t\mathbf{y}), \frac{1}{t} g((1-t)\mathbf{c} + t\mathbf{y}), t), \tag{30}$$

where we factored out the root $t = 0$ which corresponds to the trivial intersection $\mathbf{c}$. By evaluating the resultant we obtain the equation

$$k(\mathbf{y}) = cy_3^2 y_4 + by_2^2 y_4 - y_1 y_4^2 + y_1^2 y_4 + (1-b) y_1 y_2^2 + (1-c) y_1 y_3^2 \tag{31}$$

which defines a cubic surface in three-dimensional real projective space with homogeneous coordinates $y_1 : y_2 : y_3 : y_4$.

Step (3) A close inspection reveals the fact that the cubic surface $k(\mathbf{y})$ contains the straight line $\mathbf{l}(u) = (0, 1, u, 0)^\top$. Indeed, this line is the intersection of the two-dimensional tangent plane at the center $\mathbf{c}$ of the surface in five-dimensional space with the image hyperplane.

Step (4) Now we move the tangent plane of k along this line and intersect it with the cubic. This technique is closely related to one of the local parameterization techniques for cubic surfaces that were described by Szilágyi et al. (2006). In this special case the computations become much simpler, as a line on the cubic surface is known. For any value of u, the tangent plane can be parameterized by

$$\mathbf{q}(u, s_1, s_2) = \mathbf{l}(u) + s_1 \mathbf{v}_1(u) + s_2 \mathbf{v}_2(u) \tag{32}$$

where $\mathbf{v}_1 = (0, 0, 1, 0)^\top$ and $\mathbf{v}_2 = (b + cu^2, 0, 0, (c-1)u^2 + b - 1)^\top$.

Step (5) The intersection of $\mathbf{q}(u, s_1, s_2)$ with $k(\mathbf{y})$ gives the equation

$$k(\mathbf{q}(u, s_1, s_2)) = 2u(b-c)s_1 + (b-c)s_1^2 + (u^2+1)(b+cu^2)(b-1+u^2(c-1))s_2^2, \tag{33}$$

which defines a conic section in the s_1, s_2-plane. The conic-section is non-degenerate, as $b \neq c$ was assumed. We parameterize each of these conic sections by intersecting it with lines through $(s_1, s_2) = (0, 0)$, which gives

$$s_1 = \frac{1}{N} 2u(c-b), \quad s_2 = \frac{1}{N} 2uv(c-b), \quad \text{where}$$
$$N = b - c + v^2(c^2u^4 - 2u^2b - cu^2 + b^2u^2 + c^2u^6 - bu^4 - cu^6 - 2cu^4 + 2bcu^2 + 2bcu^4 - b + b^2). \tag{34}$$

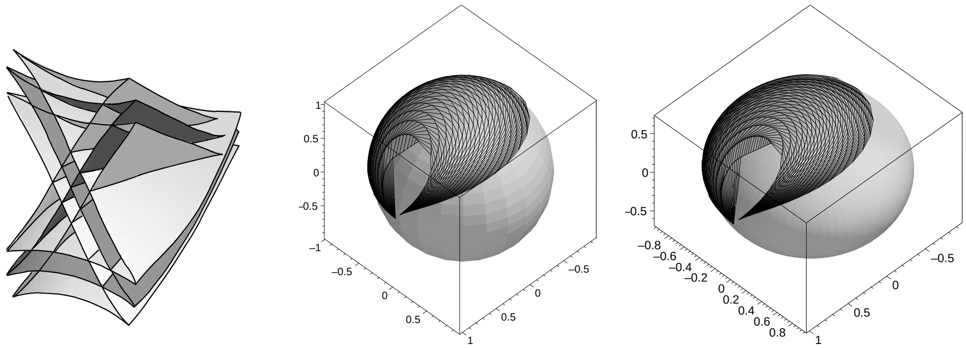
Finally we obtain a parameterization

$$\begin{aligned}
y_1 &= 2uv(b-c)(cu^2+b) \\
y_2 &= 2bcv^2u^4 + 2bcv^2u^2 + c^2v^2u^4 - cv^2u^6 + c^2v^2u^6 - bv^2 - bv^2u^4 \\
&\quad + b^2v^2u^2 - 2bv^2u^2 - 2cv^2u^4 - cv^2u^2 + b^2v^2 - c + b \\
y_3 &= u(2bcv^2u^4 + 2bcv^2u^2 + c^2v^2u^4 - v^2cu^6 + c^2v^2u^6 - bv^2 - bv^2u^4 \\
&\quad + b^2v^2u^2 - 2bv^2u^2 - 2cv^2u^4 - cv^2u^2 + b^2v^2 + c - b) \\
y_4 &= 2uv(b-c)(-1 + b + cu^2 - u^2)
\end{aligned}$$

of the cubic surface.

Step (6) We lift the parameterization back into the five-dimensional space. We substitute the parameterization $\mathbf{y}(u, v) = (y_1(u, v), y_2(u, v), y_3(u, v), y_4(u, v), 0)$ into (29) and solve this equation for $\bar{t}(u, v)$. The parameterization of the intersection is then given by

$$\mathbf{p}(u, v) = (1 - \bar{t}(u, v))\mathbf{c} + \bar{t}(u, v)\mathbf{y}(u, v). \tag{35}$$



(a) (b) (c)

Figure 2. The surface of Example 7 and its offsets (a), and the simultaneous parameterizations of the sphere and of the ellipsoid (b,c).

We apply the algorithm to two examples:

Example 7 (continued). The support function (19) fulfills the requirements of the parameterization algorithm with $b = 1$ and $c = 2$. The first stereographic projection gives the cubic surface

$$k(\mathbf{y}) = -y_1y_3^2 - y_1y_4^2 + y_4y_1^2 + y_4y_2^2 + 2y_4y_3^2. \tag{36}$$

Following the next step of the algorithm, we compute the tangent planes along the line $\mathbf{l}(u) = (0, 1, u, 0)^\top$ and intersect them with the cubic. After parameterizing them we obtain a parameterization of the cubic surface,

$$\begin{aligned} y_1 &= -2uv(1 + 2u^2) \\ y_2 &= 2v^2u^6 + 3v^2u^4 + v^2u^2 - 1 \\ y_3 &= u(2v^2u^6 + 3v^2u^4 + v^2u^2 + 1) \\ y_4 &= -2u^3v \end{aligned} \tag{37}$$

Now we can substitute these polynomials into (29) and obtain

$$\bar{t}(u, v) = \frac{4uv}{v^4(4u^{12} + 12u^{10} + 13u^8 + 6u^6 + u^4 + 1) + v^2(4u^6 + 10u^4 + 2u^2) + 1}. \tag{38}$$

Lifting this parameterization back into five-dimensional space gives the five bivariate polynomials

$$\begin{aligned} x_1 &= v^4(4u^{12} + 12u^{10} + 13u^8 + 6u^6 + u^4) + v^2(4u^6 - 6u^4 - 6u^2) + 1 \\ x_2 &= 4uv(3v^2u^4 + 2v^2u^6 + v^2u^2 - 1) \\ x_3 &= 4u^2v(3v^2u^4 + 2v^2u^6 + v^2u^2 + 1) \\ x_4 &= v^4(4u^{12} + 12u^{10} + 13u^8 + 6u^6 + u^4) + v^2(4u^6 + 2u^4 + 2u^2) + 1 \\ x_5 &= v^4(4u^{12} + 12u^{10} + 13u^8 + 6u^6 + u^4) + v^2(4u^6 + 10u^4 + 2u^2) + 1 \end{aligned} \tag{39}$$

that satisfy the two identities (25). The corresponding two parameterizations $\frac{1}{x_4}(x_1, x_2, x_3)$ and $\frac{1}{x_5}(x_1, x_2, x_3)$ of the unit sphere and of the ellipsoid are shown in Fig. 2b,c. In both cases, the parameters u, v vary in the domain $[0, 1.5] \times [0, 1.5]$. Finally we evaluate the envelope operator (26) and obtain the parameterization $\mathbf{z}(u, v)$ of the surface with the support function (19), see Fig. 2a. The parameterization is presented in Table 1.

Table 1. Parameterization of the surface of Example 7

$$
\begin{aligned}
z_1 = \frac{1}{D}&(1 - 20v^5u^6 + 32v^7u^{20} + 248v^8u^{20} + 132v^6u^{10} + 48v^5u^{12} + 36v^4u^6 + 252v^7u^{14} \\
&+ 4v^7u^8 - 72v^5u^8 + 24v^3u^8 - 12v^3u^6 + 36v^7u^{10} + 48v^5u^{14} + 4vu^2 + 360v^8u^{18} + 4v^6u^6 \\
&+ 36v^6u^8 + 8v^2u^6 + 180v^8u^{14} + 144v^7u^{18} + 264v^7u^{16} - 52v^5u^{10} + 252v^6u^{12} + 4v^2u^2 \\
&+ 72v^4u^{10} + 16v^8u^{24} + 24v^4u^{12} + 62v^8u^{12} + 62v^4u^8 + 96v^8u^{22} + v^8u^8 + 12v^8u^{10} \\
&+ 6v^4u^4 + 32v^6u^{18} - 20v^3u^4 + 144v^6u^{16} + 12v^2u^4 + 264v^6u^{14} + 321v^8u^{16} + 132v^7u^{12}) \\
&(1 + 20v^5u^6 - 32v^7u^{20} + 248v^8u^{20} + 132v^6u^{10} - 48v^5u^{12} + 36v^4u^6 - 252v^7u^{14} - 4v^7u^8 \\
&+ 72v^5u^8 - 24v^3u^8 + 12v^3u^6 - 36v^7u^{10} - 48v^5u^{14} - 4vu^2 + 360v^8u^{18} + 4v^6u^6 + 8v^2u^6 \\
&+ 36v^6u^8 + 180v^8u^{14} - 144v^7u^{18} - 264v^7u^{16} + 52v^5u^{10} + 252v^6u^{12} + 4v^2u^2 + 72v^4u^{10} \\
&+ 16v^8u^{24} + 24v^4u^{12} + 62v^8u^{12} + 62v^4u^8 + 96v^8u^{22} + 12v^8u^{10} + 6v^4u^4 + v^8u^8 \\
&+ 32v^6u^{18} + 20v^3u^4 + 144v^6u^{16} + 12v^2u^4 + 264v^6u^{14} + 321v^8u^{16} - 132v^7u^{12})
\end{aligned}
$$

$$
\begin{aligned}
z_2 = \frac{1}{D}&64u^5v^3(1 - 3v^2u^4 - 2v^2u^6 - v^2u^2)(1 + 3v^2u^4 + 2v^2u^6 + v^2u^2)^2 \\
&(4v^4u^{12} + 12v^4u^{10} + 13v^4u^8 + 4v^2u^6 + 6v^4u^6 - 6v^2u^4 + v^4u^4 - 6v^2u^2 + 1)
\end{aligned}
$$

$$
\begin{aligned}
z_3 = \frac{1}{D}&4(4v^4u^12 + 12v^4u^{10} + 13v^4u^8 + 4v^2u^6 + 6v^4u^6 - 6v^2u^4 + v^4u^4 - 6v^2u^2 + 1) \\
&u^2v(1 + 3v^2u^4 + 2v^2u^6 + v^2u^2)(1 + 12v^3u^6 + 8v^3u^8 + 4v^2u^6 + 4vu^2 + v^4u^4 + 4v^3u^4 \\
&+ 13v^4u^8 + 4v^4u^12 + 12v^4u^{10} + 2v^2u^2 + 2v^2u^4 + 6v^4u^6)(1 - 12v^3u^6 - 8v^3u^8 + 4v^2u^6 \\
&- 4vu^2 + 13v^4u^8 + 4v^4u^{12} + 12v^4u^{10} + 2v^2u^2 + 2v^2u^4 + v^4u^4 - 4v^3u^4 + 6v^4u^6)
\end{aligned}
$$

$$
\begin{aligned}
D = &(1 + 3v^2u^4 + 2v^2u^6 + v^2u^2 + 2vu^2)^3(1 + 3v^2u^4 + 2v^2u^6 + v^2u^2 - 2vu^2)^3 \\
&(1 + 2v^2u^2 + 10v^2u^4 + 4v^2u^6 + v^4u^4 + 6v^4u^6 + 13v^4u^8 + 12v^4u^{10} + 4v^4u^{12})
\end{aligned}
$$

Example 12. In this example we consider the surface given by the support function $h(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 - x_3^2} + 1$. It is the offset at distance 1 of a one-sheeted hyperboloid of revolution. Applying the parameterization process as in the previous example, we obtain the following parameterization:

$$
\begin{aligned}
z_1 &= \frac{1}{D}(2v^4u^{12} - 4v^4u^8 - 4v^2u^6 + 2v^4u^4 - 12v^2u^2 + 2)(-1 + v^2u^6 - v^2u^2)^2 \\
z_2 &= -\frac{1}{D}8uv(-1 + v^2u^6 - v^2u^2)^2(1 + v^2u^6 - v^2u^2) \\
z_3 &= \frac{1}{D}64u^6v^3(-1 + v^2u^6 - v^2u^2) \\
D &= (v^4u^{12} - 2v^4u^8 - 2v^2u^6 - 8u^4v^2 + v^4u^4 + 2v^2u^2 + 1) \\
&\quad (v^4u^{12} - 2v^4u^8 - 2v^2u^6 + 8u^4v^2 + v^4u^4 + 2v^2u^2 + 1)
\end{aligned}
$$

Although the degree of this surface is $(8, 24)$, the representation is quite compact as the polynomials are very sparse.

Lemma 10, combined with the results of this section, gives the following theorem.

Theorem 13. *For a surface with a support function of the form (14) with $b \neq c$ we obtain a piecewise rational parameterization by combining the result of Algorithm 11 with Lemma 10. If b, c and all other coefficients in the given support function h are rational numbers, then all coefficients of this parameterization are again rational.*

Consequently, since the class of support functions of the form (14) is closed with respect to addition of constants, these surfaces are a special case of surfaces with rational offsets (cf. Pottmann, 1995).

Finally we analyze the case $b = c$, which was excluded so far.

Remark 14. If $b = c = 1$, then we can simply parameterize the unit sphere and choose $x_4 = \pm x_5$. If $1 \neq b = c > 0$ we can solve the problem by swapping the first two coordinates.

The case $b = c < 0$ is more involved. For instance, the offsets of two-sheeted hyperboloids of revolution belong to this case. After stereographic projection and dehomogenization ($x_4 = 1$) we obtain the cubic surface

$$k = (x_1 + b - bx_1)x_2^2 + (x_1 + b - bx_1)x_3^2 + x_1^2 - x_1 = 0 \tag{40}$$

which can be rewritten as

$$r_1 r_2 x_2^2 + r_1 r_2 x_3^2 = r_2^2, \tag{41}$$

with $r_1 = (x_1 + b - bx_1)$ and $r_2 = x_1 - x_1^2$. In order to admit solutions, the factor $r_1 r_2$ has to be positive. This is the case if $x_1 \in \,]-\infty, 0[$ or $x_1 \in \,]\frac{-b}{1-b}, 1]$. Here we discuss the first situation. The second one can be treated similarly.

After substituting $x_1 = -t^2$ in (40), we obtain

$$r_1 r_2 = t^6 + t^4 - 2bt^4 - bt^2 - bt^6 = A^2 + B^2 \quad \text{and} \quad r_2 = -t^4 - t^2. \tag{42}$$

with

$$A = t^3\sqrt{1-b} - t\sqrt{-b} \quad \text{and} \quad B = t^2\sqrt{1-b} + t^2\sqrt{-b}. \tag{43}$$

Note that $r_1 r_2$ is now non–negative for all values of t, hence it is possible to represent it as a sum of two squares. The point with coordinates

$$x_1 = -t^2, \quad x_2 = \frac{A r_2}{A^2 + B^2} \quad \text{and} \quad x_3 = \frac{B r_2}{A^2 + B^2} \tag{44}$$

lies on each of the circles, and it can be used to create a parameterization of the cubic surface (40). Note that this parameterization has coefficients involving certain square roots of the original coefficients, as a decomposition of a polynomial into a sum of squares is needed.

6. Conclusion

Motivated by the analysis of offsets to quadric surfaces, we analyzed a class of surfaces which have special support functions of the form (14). It was shown that the surfaces of this class, which is closed under offsetting, admit rational parameterizations. Hence they are special instances of surfaces with rational offsets, which were discussed by Pottmann (1995). On the other hand, this class of surfaces comprises both surfaces with rational support functions and quadric surfaces.

We show that the rational parameterization of surfaces from this class is closely related to the parameterization of del Pezzo surfaces. If the given support function involves only coefficients which are rational numbers, then the coefficients of these parameterizations are again rational numbers.

In particular, this relation to del Pezzo surface exists for offsets of quadric surfaces. In that case, our method produces a parameterization of higher degree than the parameterization described by Peternell and Pottmann (1998). The technique described in the present paper is more general, as it can deal with a larger class of surfaces. As a potential advantage, it relies solely on rational operations. In particular, no decomposition of a non-negative polynomial in a sum of squares – hence no field extension – is required.

As a possible topic of future work one may look into general rational parameterizations of the cubic surfaces from the previous sections. It can be shown that each rational parameterization of a surface with a support function (14) corresponds to a rational parameterization of this cubic. Consequently, one may try to obtain parameterizations of lower by using other parameterizations of the cubic surfaces. In addition, methods for obtaining proper parameterizations would be of potential interest.

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