Surfaces with rational chord length parameterization

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Abstract. We consider a rational triangular Bézier surface of degree n and study conditions under which it is rationally parameterized by chord lengths (RCL surface) with respect to the reference circle. The distinguishing property of these surfaces is that the ratios of the three distances of a point to the three vertices of an arbitrary triangle inscribed to the reference circle and the ratios of the distances of the parameter point to the three vertices of the corresponding domain triangle are identical. This RCL property, which extends an observation from [10, 13] about rational curves parameterized by chord lengths, was firstly observed in the surface case for patches on spheres in [2]. In the present paper, we analyze the entire family of RCL surfaces, provide their general parameterization and thoroughly investigate their properties.

1 Introduction

Recently, chord length parametrization has become an active research area in Computer Aided Geometric Design. This approach was motivated by the use of chord length parameterization for interpolation and approximation of discrete point data. It can be seen as an alternative to arc-length parameterizations because analogously to arc-length parameter, the chord-length parameter is also uniquely given by the loci of the curve.

A geometric proof of the fact that rational quadratic circles in standard Bézier form are parameterized by chord–length was done in [11]. An alternative proof by Mathematica can be found in [6]. A thorough analysis followed in [10, 13], where two independent constructions for general rational curves of this type were presented. In some sense, rational curves with chord length parameterizations (shortly RCL curves) are a chord-length analogy to the so called Pythagorean-hodograph curves characterized by closed form expressions for their arc-lengths, cf. [7, 9].

Curves with RCL property are worth studying mainly because of the following advantages. First they provide a simple inversion formula, which can be e.g. used for computing their implicit equations. Second, it is simple to perform point-curve testing. Finally, these curves do not possess self-intersections. In addition to straight lines and circles in standard form, this class of RCL curves also contain e.g. equilateral hyperbola, Bernoulli's lemniscate and Pascal's Limaçon. Curves with chord-length parameterization were also mentioned among remarkable families of curves admitting a complex rational form in [12].

Motivated by RCL curves, it is natural to extend this approach also to rational surfaces. A promising result was presented in [2] where the equal chord property of quadratic rational Bézier patches describing a segment of a sphere was proved. For this, the well-known construction of spherical quadratic patches by stereographic projection was used, cf. [1, 4, 5]. This result directly extends the planar result for circles, see [6]. As a byproduct, it was shown in [2] how to characterize this property using tripolar coordinates in space, which extend the observations of [13] concerning the relation between bipolar coordinates (see [3, 8] for more details) and curves with chord-length parameterization.

The present paper is devoted to the equal chord property of rational triangular Bézier surfaces of degree n, thus extending the results of [10, 13] to the case of surfaces. We present a general construction of rational chord length parameterizations (RCL surfaces) and study their attractive geometric properties. The introduced approach is then demonstrated by several examples of RCL surfaces.

2 Preliminaries

We consider a rational surface of degree n, which is described by its triangular Bernstein–Bézier representation

$$\mathbf{P}(\mathbf{X}) = \frac{\displaystyle\sum_{i,j,k\in\mathbb{Z}^+,\ i+j+k=n} w_{ijk}\,\mathbf{b}_{ijk}\,\frac{n!}{i!j!k!}\lambda^i\mu^j\nu^k}{\displaystyle\sum_{i,j,k\in\mathbb{Z}^+,\ i+j+k=n} w_{ijk}\,\frac{n!}{i!j!k!}\lambda^i\mu^j\nu^k}, \quad \mathbf{X}\in\mathbb{R}^2 \tag{1}$$

with respect to a non-degenerate reference triangle $\triangle(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)\subset\mathbb{R}^2$ with vertices $(\mathbf{A}_\ell)_{\ell=1,2,3}$. Its argument

$$\mathbf{X} = \lambda\mathbf{A}_1 + \mu\mathbf{A}_2 + \nu\mathbf{A}_3, \quad \lambda+\mu+\nu = 1, \tag{2}$$

is expressed by barycentric coordinates with respect to the reference triangle.

The shape of the surface is determined by the $\binom{n+1}{2}$ control points $\mathbf{b}_{ijk}$ with the associated weights w_{ijk}. In particular, the control net of the patch has the three vertices

$$\mathbf{v}_1 = \mathbf{b}_{n00}, \quad \mathbf{v}_2 = \mathbf{b}_{0n0}, \quad \text{and} \quad \mathbf{v}_3 = \mathbf{b}_{00n} \tag{3}$$

which are the images of the vertices of the reference triangle.

Let

$$R_\ell(\mathbf{X}) = ||\mathbf{X}-\mathbf{A}_\ell||^2 \quad \text{and} \quad r_\ell(\mathbf{X}) = ||\mathbf{P}(\mathbf{X})-\mathbf{v}_\ell||^2 \tag{4}$$

be the squared distances of the point $\mathbf{X}$ and its image $\mathbf{P}(\mathbf{X})$ to the vertices of the domain triangle and to the vertices of the patch, respectively.

Definition 1. *The surface* (1) *is a rational chord length parameterization (RCL) with respect to the reference triangle, if*

$$r_1 : r_2 : r_3 = R_1 : R_2 : R_3, \quad \textit{or, equivalently,}$$
$$\forall (i,j) \in \{(1,2),(2,3),(3,1)\} : \quad r_i R_j = r_j R_i \tag{5}$$

holds for all points $\mathbf{X} \in \mathbb{R}^2$.

We first analyze the relation between the reference triangle and the triangle spanned by the vertices of the control net.

Lemma 1. *If the surface is a rational chord length parameterization, then the triangles* $\triangle(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$ *and* $\triangle(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ *are similar.*

Proof. We evaluate the three relations (5) at the three vertices $\mathbf{A}_\ell$ of the domain triangle. Six of these 9 equations are trivially satisfied, since one of the r_i and R_i vanish at each vertex. The remaining three equations guarantee the similarity of the triangles. □

In the remainder of the paper, we identify the reference triangle $\triangle(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$ with the vertex triangle $\triangle(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and the domain $\mathbb{R}^2$ containing it with the plane spanned by the vertex triangle. Consequently, *the domain of the surface is the plane spanned by the vertex triangle.*

For any point $\mathbf{Y} \in \mathbb{R}^3$, we denote with

$$\varrho_\ell(\mathbf{Y}) = ||\mathbf{Y} - \mathbf{v}_\ell||^2, \quad \ell = 1,2,3, \tag{6}$$

the squared distances to the vertices of the patch.

Lemma 2. *The set of all points* $\mathbf{Y}$ *satisfying*

$$\forall (i,j) \in \{(1,2),(2,3),(3,1)\} : \quad \varrho_i(\mathbf{Y}) R_j(\mathbf{X}) = \varrho_j(\mathbf{Y}) R_i(\mathbf{X}) \tag{7}$$

is a circle which passes through $\mathbf{X}$ *and is perpendicular to any sphere containing the vertices of the patch. If* $\mathbf{X}$ *lies on the circumcircle of the vertex triangle, then the circle* $\mathbf{Y}$ *shrinks to the single point* $\mathbf{X}$.

Proof. Recall that for any two points $\mathbf{M}$, $\mathbf{N}$ in the plane, the set of all points $\mathbf{Z}$ satisfying

$$||\mathbf{Z} - \mathbf{M}||^2 = c\,||\mathbf{Z} - \mathbf{N}||^2 \tag{8}$$

for some positive constant c is a circle (Apollonius' definition) which intersects any circle through $\mathbf{M}$ and $\mathbf{N}$ orthogonally. Consequently, for a given point $\mathbf{X}$, the set of all points $\mathbf{Y}$ satisfying

$$\varrho_i(\mathbf{Y}) R_j(\mathbf{X}) = \varrho_j(\mathbf{Y}) R_i(\mathbf{X}) \tag{9}$$

is a sphere whose center lies on the line through $\mathbf{v}_i$ and $\mathbf{v}_j$. Moreover, any sphere containing these two vertices intersects this sphere orthogonally. Indeed, if we consider the intersection with the common symmetry plane of both spheres, which is spanned by the sphere's center and the line through $\mathbf{v}_i$ and $\mathbf{v}_j$, then we obtain the two families of circles which appear in Apollonius' definition of a circle.

Clearly, the three spheres (9) obtained for $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$ intersect in one circle, since the equations defining them are not independent. Moreover, since these spheres intersect any sphere through the three points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, orthogonally, so does the intersection curve, cf. Fig. 1.

If $\mathbf{X}$ belongs to the circumcircle of the vertex triangle, then any two of the three spheres (9) touch each other at this point and the circle degenerates into a single point. □

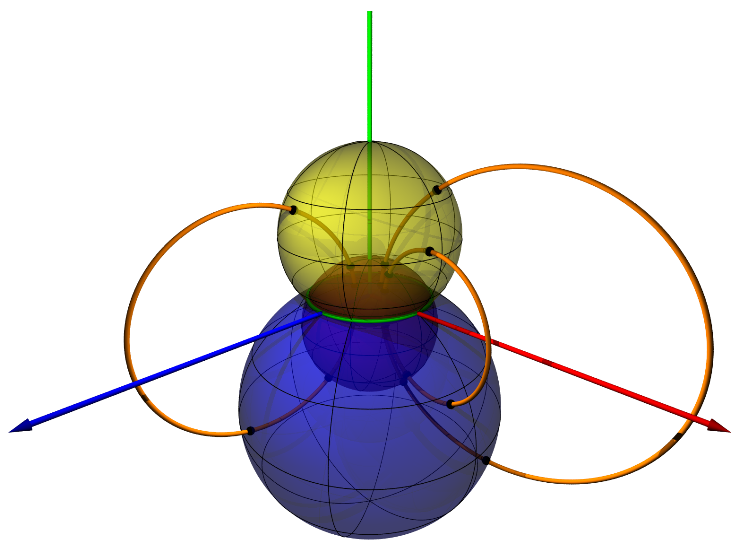

Fig. 1. Examples of circles perpendicular to any sphere containing the reference circle.

Corollary 1. *If* $\mathbf{P}$ *is a rational chord length parameterization, then its restriction to the circumcircle of the reference triangle is the identity. Moreover, the surface is a rational chord length parameterization with respect to any reference triangle which possesses the same circumcircle.*

Proof. The surface $\mathbf{P}$ is a RCL surface if and only if any point $\mathbf{P}(\mathbf{X})$ lies on the circle described in Lemma 2. On the one hand, if $\mathbf{X}$ is on the circumcircle of the reference triangle, then this circle shrinks to the point $\mathbf{X}$ itself. On the other hand, the family of circles described in Lemma 2 does not depend on choice of the reference triangle. □

Consequently, the RCL surface always contains the circumcircle of its reference triangle, and its definition depends solely on this circle. The latter fact can also be concluded from Corollary 4 of [2]. This observation motivates the following extended definition.

Definition 2. *A surface $\mathbf{P}$ is said to be a rational chord length parameterization with respect to a circle, if it is a rational chord length parameterization with respect to a reference triangle possessing this circle as its circumcircle.*

3 Construction of RCL surfaces

In order to simplify the formulas, we choose the reference circle as the unit circle $\mathcal{C}$ in the xy-plane. Consequently, the arguments of the rational surface $\mathbf{P}$ are all points of the form $\mathbf{X} = (u, v, 0)^\top$.

Theorem 1. *A surface $\mathbf{P}$ is a rational chord length parameterization with respect to the reference circle $\mathcal{C}$ if and only if there exists a rational function $q : (u, v) \mapsto q(u, v)$ such that*

$$\mathbf{P}(u,v) = \left(\frac{(1+q^2)u}{1+q^2(u^2+v^2)}, \frac{(1+q^2)v}{1+q^2(u^2+v^2)}, \frac{q(1-u^2-v^2)}{1+q^2(u^2+v^2)} \right)^\top . \tag{10}$$

Proof. Without loss of generality, we consider the reference triangle with the vertices $\mathbf{A}_1 = (1, 0, 0)^\top$, $\mathbf{A}_2 = (0, 1, 0)^\top$, $\mathbf{A}_3 = (0, -1, 0)^\top$ on the reference circle $\mathcal{C}$. The surface $\mathbf{P}$ is RCL if and only if there exists a rational function λ such that the squared distances R_ℓ and r_ℓ are related by

$$\forall (u,v) : \ \lambda(u,v) R_\ell(u,v) = r_\ell(u,v), \quad \ell = 1,2,3. \tag{11}$$

A short computation confirms that the intersection points of the three spheres with centers $\mathbf{A}_i$ and radii $\sqrt{r_i}$ has the coordinates

$$\begin{aligned} \mathbf{P}_\pm(u,v) = \frac{1}{4}\big(& -2r_1 + r_2 + r_3, -r_2 + r_3, \\ & \pm\sqrt{2} \cdot \sqrt{4(r_2+r_3) - [(r_1-r_2)^2 + (r_1-r_3)^2] - 8} \ \big)^\top . \end{aligned} \tag{12}$$

Using (11) and the identities $R_1 = (u-1)^2 + v^2$, $R_2 = u^2 + (v-1)^2$, $R_3 = u^2 + (v+1)^2$, which follow from the definition (4), this can be rewritten as

$$\mathbf{P}_\pm(u,v) = \big(\lambda u, \lambda v, \pm\sqrt{(1-\lambda)(\lambda u^2 + \lambda v^2 - 1)} \ \big)^\top . \tag{13}$$

This surface has a rational parameterization with respect to u, v if and only if the argument of the square root is a perfect square. This is equivalent to the condition on the existence of a rational function $q(u, v)$ such that

$$1 - \lambda = q^2(\lambda u^2 + \lambda v^2 - 1). \tag{14}$$

Solving (14) for λ we arrive at

$$\lambda(u,v) = \frac{1+q(u,v)^2}{1+q(u,v)^2(u^2+v^2)}. \tag{15}$$

Finally, we substitute λ into (13). The two possible choices of the sign of the third coordinate can be obtained by specifying the sign of the rational function q. □

We provide a geometric meaning for this result.

Proposition 1. *Consider the angle $\alpha(u,v) \in [-\pi,\pi]$ which satisfies*

$$\tan\frac{\alpha(u,v)}{2} = q(u,v). \tag{16}$$

If $u^2+v^2 \neq 1$, then α is the angle between the xy-plane and the sphere which passes through the point $\mathbf{P}(u,v)$ and the reference circle $\mathcal{C}$. If $u^2+v^2=1$, then $\mathbf{P}(u,v)$ lies on the reference circle $\mathcal{C}$ and α is the angle between the xy-plane and the tangent plane of the surface $\mathbf{P}$ at this point.

Proof. We consider a surface (10). In the first case, the unique sphere which passes through the reference circle and through the point $\mathbf{P}(u,v)$ has the center $\mathbf{C} = (0,0,(q^2-1)/(2q))^\top$ and the radius $r = (q^2+1)/(2|q|)$. The oriented angle α between the sphere and the xy-plane is equal to the angle between the vectors $(\mathbf{C}-\mathbf{A}_1)$ and $(0,0,1)^\top$, which gives $\tan\alpha = \frac{2q}{1-q^2}$. The second case can be proved similarly by a direct computation. □

Remark 1. The angle α is equal to the angle which is used in the definition of tripolar coordinates, as introduced in [2].

The following observation provides an alternative geometric interpretation of the characterization result (10).

Proposition 2. *Any RCL surface* (10) *with the reference circle $\mathcal{C}$ can be obtained by composing*

(i) the inversion M with respect to the sphere centered at $(0,-1,0)^\top$ with radius $\sqrt{2}$,
(ii) the rotation R_α about the x-axis through the angle $\alpha(u,v)$, where q satisfies (16)*, and*
(iii) the same inversion as in (i) ,

and applying this transformation to the parameterization $(u,v,0)^\top$ of the plane containing $\mathcal{C}$.

Proof. The rotation (ii) and the inversion (i,iii) are described by

$$R_\alpha(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-q^2}{1+q^2} & -\frac{2q}{1+q^2} \\ 0 & \frac{2q}{1+q^2} & \frac{1-q^2}{1+q^2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{17}$$

and

$$M(x,y,z) = \frac{1}{x^2+(y+1)^2+z^2}\begin{pmatrix} 2x \\ 1-x^2-y^2-z^2 \\ 2z \end{pmatrix}. \tag{18}$$

A direct computation now confirms that

$$\mathbf{P}(u,v) = (M \circ R_\alpha \circ M)(u,v,0), \tag{19}$$

cf. (16) and (10). □

Remark 2. The characterization (19) of RCL surfaces can be derived directly, as follows. The inversion M maps the reference circle to the x-axis and the circles of constant chord-length ratios described in Lemma 2 to coaxial circles around it. Consequently, $M(\mathbf{P}(u,v))$ can be obtained by applying the rotation R_α to the point $M(u,v,0)$. This leads to (19), since $M = M^{-1}$. All RCL surfaces can be obtained in this way, since M is a birational mapping. Proposition 1 can also be derived from this construction, since the spherical inversion M is a conformal transformation.

4 Properties and examples of RCL surfaces

In this section we will review some attractive properties of RCL surfaces and demonstrate them on some interesting examples which are computed using (10) for different choices of $q(u,v)$. Obviously, by choosing a constant function $q(u,v)$, we obtain a sphere, cf. [2].

Proposition 3. *Any RCL surface $\mathbf{P}(u,v)$ has a rational unit normal field along the reference circle. On the other hand, any rational unit normal field along the reference circle can be extended to an RCL surface. Finally, two RCL surfaces given by* (10) *with functions q_1, q_2 have the same normals along the reference circle if and only if*

$$q_1 - q_2 = (1-u^2-v^2)f, \tag{20}$$

where $f(u,v)$ is a rational function.

Proof. Under the condition $u^2+v^2=1$, the unit normal of $\mathbf{P}$ can be computed from (10) as

$$\left(\frac{2qu}{1+q^2}, \frac{2qv}{1+q^2}, \frac{1-q^2}{1+q^2}\right)^\top. \tag{21}$$

This gives also the second statement. Finally, the third part is a direct consequence. □

Let I denotes the circle inversion with respect to the reference circle in the u,v plane, i.e.,

$$I(u,v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)^\top.$$

The following proposition can be verified by a straightforward computation.

Proposition 4. *The two surfaces* $\mathbf{P}_1(u,v)$, $\mathbf{P}_2(u,v)$ *obtained for* $q(u,v)$ *and* $-1/q(I(u,v))$, *respectively, are identical up to the reparameterization via* I, *i.e.,*

$$\mathbf{P}_1(u,v) = \mathbf{P}_2(I(u,v)).$$

Definition 3. *For a given* q *let us call the restriction of* $\mathbf{P}_1(u,v)$, *or* $\mathbf{P}_2(u,v)$ *to the reference disc (i.e., to the interior of the reference circle) the* first branch, *or the* second branch *of the associated RCL surface.*

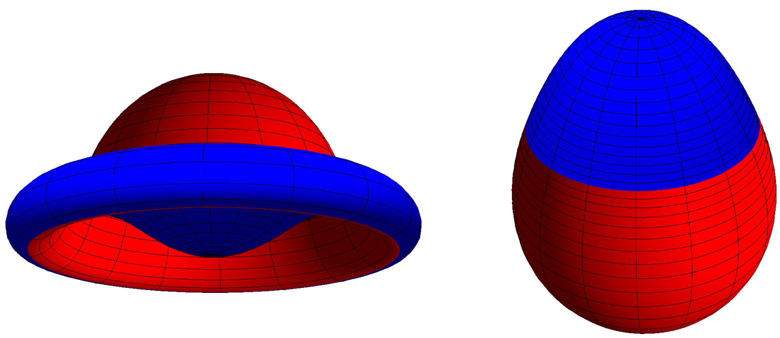

Fig. 2. Left: $q(u,v) = 1 - u^2 - v^2$; Right: $q(u,v) = 1/(1 + u^2 + v^2) + 1$.

Figures 2 and 3 (left) present examples of the surfaces mentioned in Proposition 4, where red and blue patches correspond to $\mathbf{P}_1(u,v)$ and $\mathbf{P}_2(u,v)$, respectively.

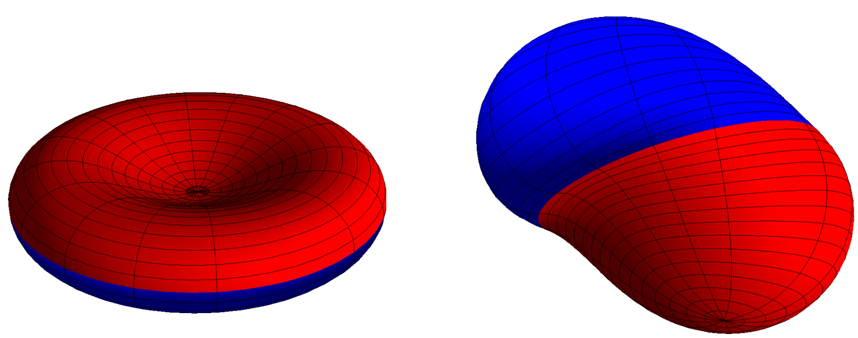

Fig. 3. Left: $q(u,v) = u^2 + v^2$; Right: $q(u,v) = u + 1$.

Proposition 5. *If $q(u,v)$ (or $1/q(I(u,v))$) does not possess a pole at $(0,0)$, then the first branch (or the second branch) is smooth and bounded. In particular, if both these conditions hold, then the entire RCL surface is a closed bounded smooth surface.*

Proof. If there is no pole for q at $(0,0)$, then $\mathbf{P}_q(0,0) = (0,0,q)^\top$ is well defined and finite. By a continuity argument the same holds for some neighborhood of $(0,0)$. The remainder of the first branch is also bounded, since each point must lie on the corresponding circle – see Lemma 2. The same argument holds for the second branch and $-1/q(I(0,0))$. □

Proposition 6. *The first branches $\mathbf{P}_1$, $\tilde{\mathbf{P}}_1$ of two RCL surfaces join with G^1 continuity along the reference circle if and only if $q\tilde{q} = -1$ for $u^2 + v^2 = 1$.*

Proof. The first branches $\mathbf{P}_1$ and $\tilde{\mathbf{P}}_1$ join with G^1 continuity along the reference circle iff $\tilde{\alpha} = -(180^\circ - \alpha)$. Hence, $\tilde{q} = \tan\frac{\tilde{\alpha}}{2} = -\tan\left(90^\circ - \frac{\alpha}{2}\right) = -\cot\frac{\alpha}{2} = -1/q$. □

Figures 3 (right) and 4 show examples of surfaces described in Proposition 6, where red and blue patches correspond to $\mathbf{P}_1$ and $\tilde{\mathbf{P}}_1$, respectively.

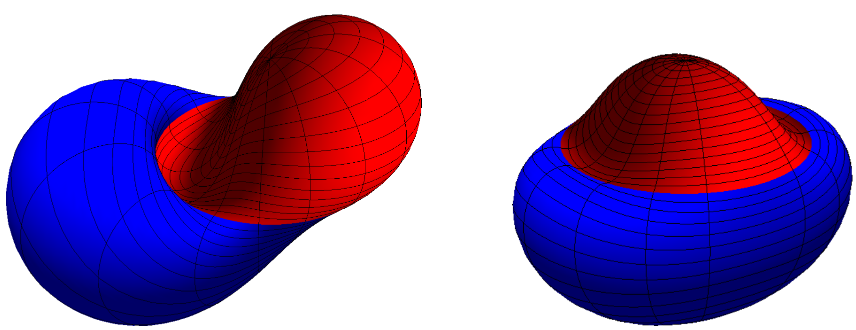

Fig. 4. Left: $q(u,v) = 2u + v + 1$; Right: $q(u,v) = u^2 - 2/3$.

5 Conclusion

We described a class of rational triangular Bézier surfaces possessing a parameterization which preserves the distance ratios to the vertices of the domain triangle inscribed to the reference circle. This extends the property of chord-length parameterization of rational curves, which was studied in [10] and [13], to the case of surfaces. We identified a family of RCL surfaces, characterized their general parameterization and studied their properties. The future research will be focused mainly on modeling with surface patches of this type.

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