Curves and surfaces with rational chord length parameterization

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Abstract

The investigation of rational varieties with chord length parameterization (shortly RCL varieties) was started by Farin (2006) who observed that rational quadratic circles in standard Bézier form are parametrized by chord length. Motivated by this observation, general RCL curves were studied. Later, the RCL property was extended to rational triangular Bézier surfaces of an arbitrary degree for which the distinguishing property is that the ratios of the three distances of a point to the three vertices of an arbitrary triangle inscribed to the reference circle and the ratios of the distances of the parameter point to the three vertices of the corresponding domain triangle are identical. In this paper, after discussing rational tensor-product surfaces with the RCL property, we present a general unifying approach and study the conditions under which a k-dimensional rational variety in d-dimensional Euclidean space possesses the RCL property. We analyze the entire family of RCL varieties, provide their general parameterization and thoroughly investigate their properties. Finally, the previous observations for curves and surfaces are presented as special cases of the introduced unifying approach.

Key words: rational varieties, chord lengths parameterizations, rational Bézier patches

1 Introduction

The investigation of rational varieties with chord length parameterization (shortly RCL varieties) started in Farin (2006) for planar curves by showing that the chord length parameter assignment is exact for circle segments

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in standard rational quadratic form. An independent geometric proof of this fact can be found also in Sabin and Dodgson (2005), where an application to a circle-preserving variant of the four-point subdivision scheme is discussed.

Investigating chord length parameterizations in Computer Aided Geometric Design is mainly motivated by the use of chord length or chordal method for interpolation and approximation of discrete point data. RCL parameterizations can be seen as an alternative to arc-length parameterizations because – analogously to an arc-length parameter – the chord-length parameter is also uniquely determined by the loci of the curve. Hence, rational curves with chord length parameterizations are, in some sense, a chord-length analogy to the so called Pythagorean-hodograph curves characterized by closed form formulas for their arc-lengths, cf. Farouki (2008); Farouki and Sakkalis (1990). Let us emphasize that RCL curves are worth studying mainly because of the following advantages: they provide a simple inversion formula applicable e.g. for computing their implicit form, they do not possess self-intersections, and they are suitable for point-curve testing.

Motivated by the recent work, a thorough analysis of curves with RCL property followed. Sánchez-Reyes and Fernández-Jambrina (2008) studied the close connection between bipolar coordinates and curves with chord length parametrization. It was shown that RCL curves are simply those whose parameter coincides with one of the bipolar coordinates. Independently, Lü (2009) uses computations in the complex plane for studying rational curves which can be parametrized by chord length and he presents two schemes characterizing planar and spatial curves. Among other results, cubics and quartics are studied and consequently applied to geometric Hermite interpolation. Finally, some curves with chord-length parameterization were mentioned among remarkable curves possessing a complex rational form, see Sánchez-Reyes (2009). Besides straight lines and circles in standard form, the family of RCL curves contains e.g. equilateral hyperbolas, Bernoulli's lemniscate and Pascal's Limaçon.

Promising observations concerning RCL curves motivated to extend this approach to rational surfaces. First, it was proved in Bastl et al. (2011) that the equal chord property holds for certain quadratic rational Bézier patches describing a segment of a sphere. This result is a direct surface analogy to the planar result of Farin (2006). The proof is based on the well-known construction of spherical quadratic patches by stereographic projection, cf. Albrecht (1998); Dietz et al. (1993); Farin (2002). In addition, it was shown how to characterize the RCL property of surfaces using tripolar coordinates in space, which extend the results of Sánchez-Reyes and Fernández-Jambrina (2008) concerning the bipolar coordinates (see also Bateman (1938); Farouki and Moon (2000)).

A thorough analysis of surfaces with RCL property was provided in Bastl et al. (2010). Rational triangular Bézier surfaces of an arbitrary degree were considered and conditions under which they are rationally parametrized by chord lengths with respect to the reference circle were investigated. Let us recall that the distinguishing property of the RCL surfaces is that the ratios of the three distances of a point to the three vertices of an arbitrary triangle inscribed to the reference circle and the ratios of the distances of the parameter point to the three vertices of the corresponding domain triangle are identical. Attractive geometric properties of the RCL surfaces were demonstrated and several examples were presented.

The main aim of this paper is to extend the RCL property to general k-dimensional rational varieties and thus to formulate a general approach to chord length parameterizations in any dimension. First, we recall the well-known results concerning the RCL surfaces and present them in the case of tensor-product surfaces. Based on these motivating results, we will give the general definition and present some results necessary for a construction of arbitrary rational chord length varieties. A crucial result is deriving a general formula for any k-dimensional RCL variety in d-dimensional Euclidean space. Finally, the observations from Bastl et al. (2011, 2010); Farin (2006); Lü (2009); Sánchez-Reyes and Fernández-Jambrina (2008) are then identified as special instances of the results provided by the general approach.

2 Preliminaries

We consider a rational tensor-product surface of degree (m, n), which is described by its Bernstein–Bézier representation

$$\mathbf{P}(\mathbf{X}) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}\,\mathbf{b}_{ij}\,B_i^m(\lambda)B_j^n(\mu)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}\,B_i^m(\lambda)B_j^n(\mu)}, \quad \mathbf{X} \in \mathbb{R}^2, \tag{1}$$

where λ, μ are coordinates of $\mathbf{X}$ given by the bilinear parameterization

$$\mathbf{X} = \sum_{i=0}^{1}\sum_{j=0}^{1} \mathbf{A}_{ij}\,B_i^1(\lambda)B_j^1(\mu), \quad (\lambda, \mu) \in [0, 1]^2 \tag{2}$$

of a (not necessarily axis-aligned) rectangle $\square(\mathbf{A}_{00}, \mathbf{A}_{01}, \mathbf{A}_{10}, \mathbf{A}_{11})$ in the plane $\mathbb{R}^2$ with vertices $\mathbf{A}_{ij}$, which we will call the *reference rectangle*. The basis functions are the standard Bernstein polynomials.

The shape of the surface is determined by the $(m + 1)(n + 1)$ control points $\mathbf{b}_{ij}$ with the associated weights w_{ij}. In particular, the control net of the patch has the four vertices

$$\mathbf{v}_{00} = \mathbf{b}_{00}, \quad \mathbf{v}_{10} = \mathbf{b}_{m0}, \quad \mathbf{v}_{01} = \mathbf{b}_{0n}, \quad \text{and} \quad \mathbf{v}_{11} = \mathbf{b}_{mn} \tag{3}$$

which are the images of the vertices of the reference rectangle. Let

$$R_{k\ell}(\mathbf{X}) = ||\mathbf{X} - \mathbf{A}_{k\ell}||^2 \quad \text{and} \quad r_{k\ell}(\mathbf{X}) = ||\mathbf{P}(\mathbf{X}) - \mathbf{v}_{k\ell}||^2 \tag{4}$$

be the squared distances of the point $\mathbf{X}$ and its image $\mathbf{P}(\mathbf{X})$ to the vertices of the domain and to the vertices of the patch, respectively.

Definition 1 *The surface* (1) *is a rational chord length parameterization (RCL) with respect to the reference rectangle* $\square(\mathbf{A}_{00}, \mathbf{A}_{01}, \mathbf{A}_{10}, \mathbf{A}_{11})$*, if*

$$\begin{aligned} &r_{00} : r_{10} : r_{01} : r_{11} = R_{00} : R_{10} : R_{01} : R_{11}, \quad \textit{or, equivalently,} \\ &\forall (i,j),(k,\ell) \in \{(0,0),(1,0),(0,1),(1,1)\} : \quad r_{ij}R_{k\ell} = r_{k\ell}R_{ij} \end{aligned} \tag{5}$$

holds for all points $\mathbf{X} \in Q$*.*

We first analyze the relation between the reference rectangle and the quadrilateral spanned by the vertices of the control net.

Lemma 2 *If the surface is a rational chord length parameterization, then the rectangle* $\square(\mathbf{A}_{00}, \mathbf{A}_{10}, \mathbf{A}_{01}, \mathbf{A}_{11})$ *and the quadrilateral* $\square(\mathbf{v}_{00}, \mathbf{v}_{10}, \mathbf{v}_{01}, \mathbf{v}_{11})$ *are similar.*

Proof. We evaluate the six non-trivial (i.e., obtained for $(i,j) \neq (k,l)$) relations (5) at the four vertices $\mathbf{A}_\ell$ of the domain quadrilateral. 12 of these 24 equations are trivially satisfied, since one of the r_{ij} and R_{ij} vanishes at each vertex. The remaining equations guarantee the similarity of all four triangles formed by joining three vertices of each quadrilateral. This suffices to conclude that the two quadrilaterals are similar. $\square$

In the remainder of the paper, we identify the reference rectangle with vertices $\mathbf{A}_{ij}$ with the quadrilateral spanned by the vertices $\mathbf{v}_{ij}$ of the surface patch, and the domain $\mathbb{R}^2$ containing it with the plane spanned by the vertices. Consequently, *the domain of the surface patch is the plane spanned by the vertex rectangle.*

For any point $\mathbf{Y} \in \mathbb{R}^3$, we denote with

$$\varrho_{k\ell}(\mathbf{Y}) = ||\mathbf{Y} - \mathbf{v}_{k\ell}||^2, \quad k,\ell = 0,1, \tag{6}$$

the squared distances to the vertices of the patch.

Lemma 3 *For a given point* $\mathbf{X}$ *on the plane containing the reference rectangle, the set of all points* $\mathbf{Y}$ *satisfying*

$$\forall (i,j),(k,\ell) \in \{(0,0),(0,1),(1,0),(1,1)\}: \qquad \varrho_{ij}(\mathbf{Y})R_{k\ell}(\mathbf{X}) = \varrho_{k\ell}(\mathbf{Y})R_{ij}(\mathbf{X}) \tag{7}$$

is a circle which passes through $\mathbf{X}$ *and is perpendicular to any sphere containing the vertices of the patch. If* $\mathbf{X}$ *lies on the circumcircle of the vertex rectangle, then the circle* $\mathbf{Y}$ *shrinks to the single point* $\mathbf{X}$*.*

Proof. Recall that for any two points $\mathbf{M}$, $\mathbf{N}$ in the plane, the set of all points $\mathbf{Z}$ satisfying

$$||\mathbf{Z} - \mathbf{M}||^2 = c\,||\mathbf{Z} - \mathbf{N}||^2 \tag{8}$$

for some positive constant c is a circle (Apollonius' definition) which intersects any circle through $\mathbf{M}$ and $\mathbf{N}$ orthogonally. Consequently, for a given point $\mathbf{X}$, the set of all points $\mathbf{Y}$ satisfying

$$\varrho_{ij}(\mathbf{Y})R_{k\ell}(\mathbf{X}) = \varrho_{k\ell}(\mathbf{Y})R_{ij}(\mathbf{X}) \tag{9}$$

is a sphere whose center lies on the line through $\mathbf{v}_{ij}$ and $\mathbf{v}_{k\ell}$, where we assume that $(i,j) \neq (k,\ell)$. Moreover, any sphere containing these two vertices intersects this sphere orthogonally. Indeed, if we consider the intersection with the common symmetry plane of both spheres, which is spanned by the sphere's center and the line through $\mathbf{v}_{ij}$ and $\mathbf{v}_{k\ell}$, then we obtain the two families of circles which appear in Apollonius' definition of a circle.

The six spheres (9) intersect in one circle. Indeed, any triplet of spheres which are obtained by considering three of the four vertices defines a single circle, since the equations are not independent. Moreover, these circles are all identical since they pass through the given point $\mathbf{X}$ and are orthogonal to any sphere that contains the four vertices, cf. Fig. 1.

If $\mathbf{X}$ belongs to the circumcircle of the vertex quadrangle, then any two of the six spheres (9) touch each other at this point and the circle degenerates into a single point. □

So far, we considered the rational tensor-product surface (1) in Bernstein-Bézier representation with respect to the given reference rectangle in the domain (which is the entire plane $\mathbb{R}^2$). Clearly, it is possible to obtain a similar tensor-product representation as (1) for any reference rectangle. In general, however, the degree of this representation will increase to $(m+n, m+n)$. The next corollary identifies the reference rectangles where the tensor-product patch is again a rational chord length parametrization.

Corollary 4 *If* $\mathbf{P}$ *is a rational chord length parameterization, then its restriction to the circumcircle of the reference rectangle is the identity. Moreover, the surface is a rational chord length parameterization with respect to any reference rectangle that possesses the same circumcircle.*

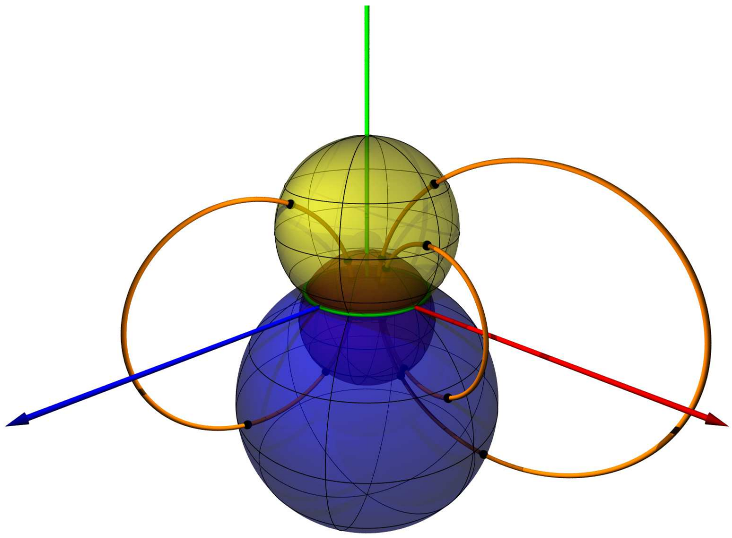

Fig. 1. Some circles which are perpendicular to any sphere containing the vertices of the patch.

Proof. The surface $\mathbf{P}$ is a RCL surface if and only if any point $\mathbf{P}(\mathbf{X})$ lies on the circle described in Lemma 3. On the one hand, if $\mathbf{X}$ is on the circumcircle of the reference rectangle, then this circle shrinks to the point $\mathbf{X}$ itself. On the other hand, the family of circles described in Lemma 3 does not depend on choice of the reference rectangle. □

Consequently, the RCL surface always contains the circumcircle of its reference rectangle, and its definition depends solely on this circle. The latter fact can also be concluded from Corollary 4 of Bastl et al. (2011). This observation motivates the following extended definition.

Definition 5 *A surface* $\mathbf{P}$ *is said to be a rational chord length parameterization with respect to a circle, if it is a rational chord length parameterization with respect to a reference rectangle possessing this circle as its circumcircle.*

In addition to considering reference rectangles, one may also consider *reference triangles*. The surface (1) can be represented as a rational triangular Bézier surface of degree $m + n$ with respect to any reference triangle. This case has been analyzed in the conference article Bastl et al. (2010). The link between chord length parameterizations with respect to rectangles and triangles is described in the following corollary, which follows immediately from the results in the previous paper.

Corollary 6 *A surface* $\mathbf{P}$ *is a rational chord length parameterization with respect to a circle (in the sense of the previous definition) if and only if any triangular Bézier surface with respect to a reference triangle that possesses the same circumcircle is a rational chord length parameterization with respect to this reference triangle (in the sense of Def. 1 in Bastl et al., 2010).*

3 Construction of RCL surfaces

In order to simplify the formulas, we choose the *reference circle* (i.e., the circumcircle of the reference rectangle) as the unit circle $\mathcal{C}$ in the xy-plane centered at the origin. Moreover, the arguments of the rational surface $\mathbf{P}$ are all points $\mathbf{X}$ of the xy-plane described by the canonical coordinates, i.e., $\mathbf{X} = (u, v, 0)^\top$.

Theorem 7 *A surface $\mathbf{P}$ is a rational chord length parameterization with respect to the reference circle $\mathcal{C}$ if and only if there exists a rational function $q : (u, v) \mapsto q(u, v)$ such that*

$$\mathbf{P}(u,v) = \left(\frac{(1+q^2)u}{1+q^2(u^2+v^2)}, \frac{(1+q^2)v}{1+q^2(u^2+v^2)}, \frac{q(1-u^2-v^2)}{1+q^2(u^2+v^2)} \right)^\top. \tag{10}$$

Proof. Without loss of generality, we consider the reference rectangle with the vertices $\mathbf{A}_{00} = (1, 0, 0)^\top$, $\mathbf{A}_{01} = (0, 1, 0)^\top$, $\mathbf{A}_{11} = (-1, 0, 0)^\top$, $\mathbf{A}_{10} = (0, -1, 0)^\top$ on the reference circle $\mathcal{C}$. The surface $\mathbf{P}$ is RCL if and only if there exists a rational function ξ such that the squared distances $R_{k,\ell}$ and $r_{k,\ell}$ are related by

$$\forall (u, v) : \ \xi(u, v) R_{k,\ell}(u, v) = r_{k,\ell}(u, v), \quad k, \ell = 0, 1. \tag{11}$$

A short computation confirms that the intersection points of the three spheres with centers $\mathbf{A}_{00}, \mathbf{A}_{01}, \mathbf{A}_{10}$ and radii $\sqrt{r_{00}}, \sqrt{r_{01}}, \sqrt{r_{10}}$ has the coordinates

$$\mathbf{P}_\pm(u,v) = \frac{1}{4} \begin{pmatrix} -2r_{00} + r_{01} + r_{10} \\ -r_{01} + r_{10}, \\ \pm\sqrt{2} \cdot \sqrt{4(r_{01}+r_{10}) - [(r_{00}-r_{01})^2 + (r_{00}-r_{10})^2] - 8} \end{pmatrix}.$$

Using (11) and the identities $R_{00} = (u-1)^2 + v^2$, $R_{10} = u^2 + (v-1)^2$, $R_{01} = u^2 + (v+1)^2$, which follow from the definition (4), this can be rewritten as

$$\mathbf{P}_\pm(u,v) = \left(\xi u, \xi v, \pm\sqrt{(1-\xi)(\xi u^2 + \xi v^2 - 1)} \right)^\top. \tag{12}$$

This surface has a rational parameterization with respect to u, v if and only if the argument of the square root is a perfect square. This is equivalent to the condition on the existence of a rational function $q(u, v)$ such that

$$1 - \xi = q^2(\xi u^2 + \xi v^2 - 1). \tag{13}$$

Solving (13) for ξ we arrive at

$$\xi(u,v) = \frac{1+q(u,v)^2}{1+q(u,v)^2(u^2+v^2)}. \tag{14}$$

Finally, we substitute ξ into (12). The two possible choices of the sign of the third coordinate can be obtained by specifying the sign of the rational function q. It can be verified that (11) is satisfied for all values of k and ℓ. □

We provide a geometric meaning for this result.

Proposition 8 *Consider the angle $\alpha(u,v) \in [-\pi, \pi]$ which satisfies*

$$\tan \frac{\alpha(u,v)}{2} = q(u,v). \tag{15}$$

If $u^2+v^2 \neq 1$, then α is the angle between the xy-plane and the sphere which passes through the point $\mathbf{P}(u,v)$ and the reference circle $\mathcal{C}$. If $u^2+v^2 = 1$, then $\mathbf{P}(u,v)$ lies on the reference circle $\mathcal{C}$ and α is the angle between the xy-plane and the tangent plane of the surface $\mathbf{P}$ at this point.

Proof. We consider a surface (10). In the first case, the unique sphere which passes through the reference circle and through the point $\mathbf{P}(u,v)$ has the center $\mathbf{C} = (0, 0, (q^2-1)/(2q))^\top$ and the radius $r = (q^2+1)/(2|q|)$. The oriented angle α between the sphere and the xy-plane is equal to the angle between the vectors $(\mathbf{C} - \mathbf{A}_1)$ and $(0,0,1)^\top$, which gives $\tan\alpha = \frac{2q}{1-q^2}$. The second case can be proved similarly by a direct computation. □

Remark 9 The angle α is equal to the angle which is used in the definition of tripolar coordinates, as introduced in Bastl et al. (2011).

The following observation provides an alternative geometric interpretation of the characterization result (10).

Proposition 10 *Any RCL surface* (10) *with the reference circle $\mathcal{C}$ can be obtained by composing*

(i) the inversion M with respect to the sphere centered at $(0,-1,0)^\top$ with radius $\sqrt{2}$,
(ii) the rotation R_α about the x-axis through the angle $\alpha(u,v)$, where q satisfies (15)*, and*
(iii) the same inversion as in (i) ,

and applying this transformation to the parameterization $(u,v,0)^\top$ of the plane containing $\mathcal{C}$.

Proof. The rotation (ii) and the inversion (i,iii) are described by

$$R_\alpha(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-q^2}{1+q^2} & -\frac{2q}{1+q^2} \\ 0 & \frac{2q}{1+q^2} & \frac{1-q^2}{1+q^2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{16}$$

and

$$M(x,y,z) = \frac{1}{x^2+(y+1)^2+z^2} \begin{pmatrix} 2x \\ 1-x^2-y^2-z^2 \\ 2z \end{pmatrix}. \tag{17}$$

A direct computation now confirms that

$$\mathbf{P}(u,v) = (M \circ R_\alpha \circ M)(u,v,0), \tag{18}$$

cf. (15) and (10). □

This geometric interpretation led us to a general approach to RCL parameterizations developed in Section 5.

4 Properties and examples of RCL surfaces

In this section we will review some attractive properties of RCL surfaces and demonstrate them on some interesting examples which are computed using (10) for different choices of $q(u,v)$. Obviously, by choosing a constant function $q(u,v)$, we obtain a sphere, cf. Bastl et al. (2011). Examples of triangular and quadrangular RCL surface patches, where vertices of the reference triangle and quadrangle, respectively, lie on the reference circle, are shown in Fig. 2.

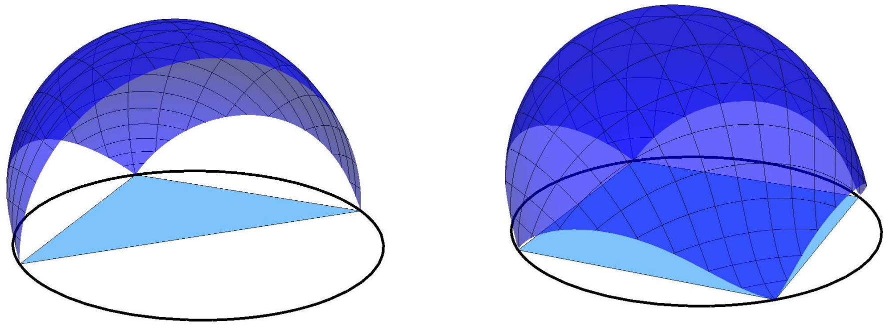

Fig. 2. Patches (triangular and quadrilateral) on RCL surface for $q(u,v) = u+1$.

The scalar-valued control points of the polynomial $q(u,v)$ can be used as design handles of RCL surfaces, both for triangular and tensor-product patches. Two examples of bi-quadratic q in Bernstein-Bézier representation with respect

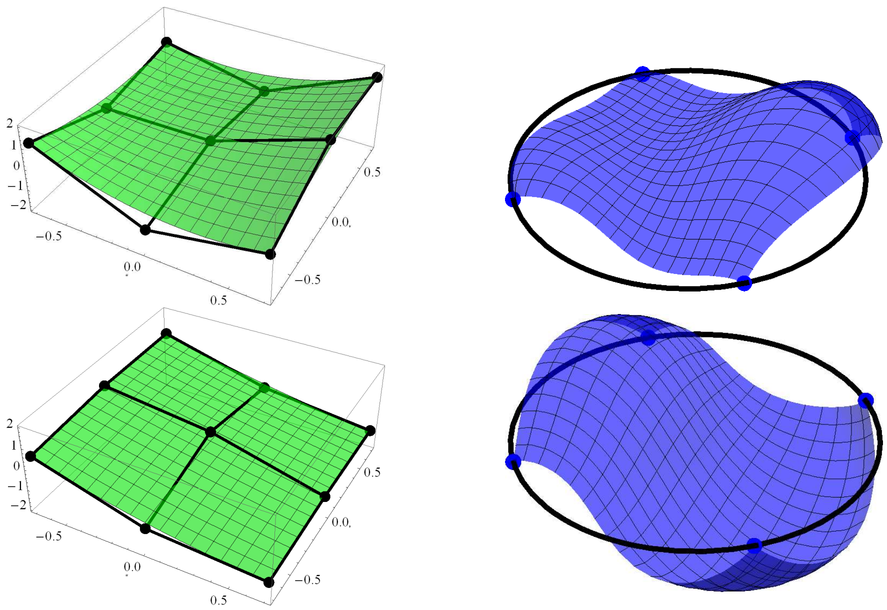

Fig. 3. Bernstein-Bézier representations of q (left) and corresponding quadrilateral RCL patches (right).

to the square with vertices $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$ and corresponding quadrilateral RCL patches are displayed in Fig. 3.

Proposition 11 *Any RCL surface $\mathbf{P}(u,v)$ has a rational unit normal field along the reference circle. On the other hand, any rational unit normal field along the reference circle can be extended to an RCL surface. Finally, two RCL surfaces given by (10) with functions q_1, q_2 have the same normals along the reference circle if and only if*

$$q_1 - q_2 = (1 - u^2 - v^2)f, \tag{19}$$

where $f(u,v)$ is a rational function.

Proof. Under the condition $u^2+v^2 = 1$, the unit normal of $\mathbf{P}$ can be computed from (10) as

$$\left(\frac{2qu}{1+q^2}, \frac{2qv}{1+q^2}, \frac{1-q^2}{1+q^2}\right)^\top. \tag{20}$$

This gives also the second statement. Finally, the third part is a direct consequence. □

Let I denote the circle inversion with respect to the reference circle in the u, v plane, i.e.,

$$I(u,v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)^\top.$$

The following proposition can be verified by a straightforward computation.

Proposition 12 *The two surfaces $\mathbf{P}_1(u,v)$, $\mathbf{P}_2(u,v)$ obtained for $q(u,v)$ and $-1/q(I(u,v))$, respectively, are identical up to the reparameterization via I,* *i.e.*,

$$\mathbf{P}_1(u,v) = \mathbf{P}_2(I(u,v)).$$

Definition 13 *For a given q let us call the restriction of $\mathbf{P}_1(u,v)$, or $\mathbf{P}_2(u,v)$ to the reference disc (i.e., to the interior of the reference circle) the* first branch, *or the* second branch *of the associated RCL surface.*

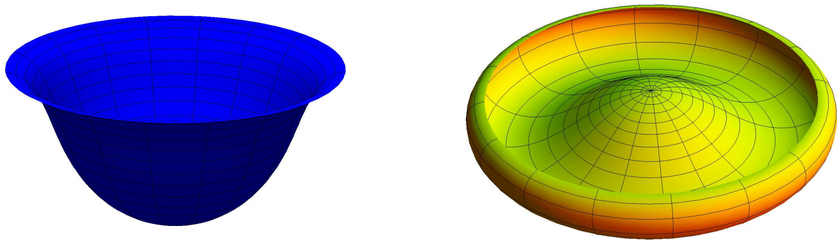

Fig. 4. RCL surface obtained for $q(u,v) = 1-u^2-v^2$. Left: First branch $\mathbf{P}_1(u,v)$; Right: Second branch $\mathbf{P}_2(u,v)$.

Fig. 4 presents an example of the two surfaces in Proposition 12, where blue and yellow patches correspond to $\mathbf{P}_1(u,v)$ and $\mathbf{P}_2(u,v)$, respectively.

Proposition 14 *If $q(u,v)$ (or $1/q(I(u,v))$) does not possess a pole at $(0,0)$, then the first branch (or the second branch) is smooth and bounded. In particular, if both conditions are satisfied, then the entire RCL surface is a closed bounded smooth surface.*

Proof. If there is no pole for q at $(0,0)$, then $\mathbf{P}_q(0,0) = (0,0,q)^\top$ is well defined and finite. By a continuity argument the same holds for some neighbourhood of $(0,0)$. The remainder of the first branch is also bounded, since each point must lie on the corresponding circle – see Lemma 3. The same argument holds for the second branch and $-1/q(I(0,0))$. □

Proposition 15 *The first branches $\mathbf{P}_1$, $\tilde{\mathbf{P}}_1$ of two RCL surfaces meet with G^1 continuity along the reference circle if and only if $q\tilde{q} = -1$ for $u^2+v^2 = 1$.*

Proof. The first branches $\mathbf{P}_1$ and $\tilde{\mathbf{P}}_1$ join with G^1 continuity along the reference circle iff $\tilde{\alpha} = -(180° - \alpha)$. Hence, $\tilde{q} = \tan\frac{\tilde{\alpha}}{2} = -\tan\left(90° - \frac{\alpha}{2}\right) = -\cot\frac{\alpha}{2} = -1/q$. □

Fig. 5 shows examples of surfaces described in Proposition 15, where blue and yellow patches correspond to $\mathbf{P}_1$ and $\tilde{\mathbf{P}}_1$, respectively.

5 General RCL varieties

We describe the general theory of k-dimensional RCL varieties in d-dimensional Euclidean space $\mathbb{E}_d$. This theory will contain all cases studied so far, in particular planar and spatial curves and surfaces.

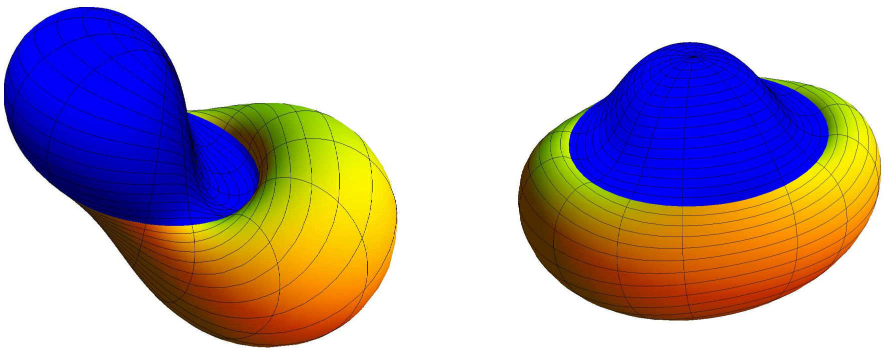

Fig. 5. Left: $q(u,v) = 2u + v + 1$; Right: $q(u,v) = u^2 - 2/3$.

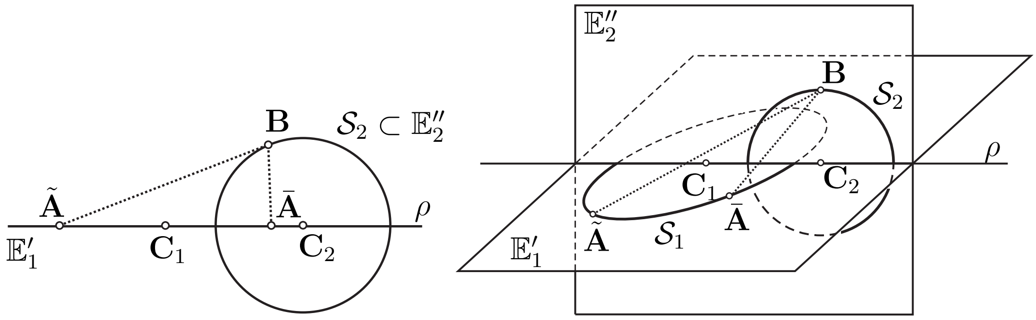

Fig. 6. Two CL-related spheres $\mathcal{S}_1 \subset \mathbb{E}'_k$ and $\mathcal{S}_2 \subset \mathbb{E}''_\ell$ in the Euclidean space $\mathbb{E}_d$ for $d = 2$, $k = 1$, $\ell = 2$ (left), and for $d = 3$, $k = 2$, $\ell = 2$ (right).

5.1 General results

Definition 16 *In a d-dimensional Euclidean space $\mathbb{E}_d$, we say that two spheres $\mathcal{S}_1(\mathbf{C}_1; r_1)$, $\mathcal{S}_2(\mathbf{C}_2; r_2)$ of dimensions $(k-1)$, $(\ell-1)$ are called chord-length related, shortly CL-related, if the following conditions are satisfied:*

(1) $k + \ell = d + 1$.
(2) The two Euclidean subspaces $\mathbb{E}'_k$ and $\mathbb{E}''_\ell$ of dimensions k, ℓ that the two spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ span, respectively, intersect in a single line ρ and are perpendicular to each other (we consider $\mathbb{E}_d$ to be perpendicular to any subspace);
(3) the two centers $\mathbf{C}_1$, $\mathbf{C}_2$ lies on the line ρ;
(4) the radii satisfy $r_1^2 + r_2^2 = |\mathbf{C}_1 - \mathbf{C}_2|^2$.

Proposition 17 *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be two CL-related spheres. Then the points of $\mathcal{S}_1$ have constant chord length ratios to the points of $\mathcal{S}_2$, and vice versa. More precisely, for any two fixed points $\tilde{\mathbf{B}}, \bar{\mathbf{B}} \in \mathcal{S}_2$ the ratio*

$$|\mathbf{A} - \tilde{\mathbf{B}}| : |\mathbf{A} - \bar{\mathbf{B}}| \tag{21}$$

is the same for all points $\mathbf{A} \in \mathcal{S}_1$ and for any two fixed points $\tilde{\mathbf{A}}, \bar{\mathbf{A}} \in \mathcal{S}_1$ the *ratio*

$$|\mathbf{B} - \tilde{\mathbf{A}}| : |\mathbf{B} - \bar{\mathbf{A}}| \tag{22}$$

is the same for all points $\mathbf{B} \in \mathcal{S}_2$*.*

Proof. By assuming a suitable choice of the coordinate system we can suppose that

$$\begin{aligned} \mathbb{E}'_k &= \{(x_1, \dots, x_d) : x_{k+1} = x_{k+2} = \cdots = x_d = 0\}, && (23) \\ \mathbb{E}''_l &= \{(x_1, \dots, x_d) : x_2 = x_3 = \cdots = x_k = 0\}, && (24) \end{aligned}$$

ρ is the x_1-axis, $\mathbf{C}_1 = (0, \dots, 0)^\top$ and $\mathbf{C}_2 = (c, 0, \dots, 0)^\top$, where $c = \sqrt{r_1^2 + r_2^2}$. For any two points

$$\begin{aligned} \mathbf{A} &= (a_1, a_2, \dots, a_k, 0, \dots, 0)^\top \in \mathcal{S}_1, && (25) \\ \mathbf{B} &= (b_1, 0, \dots, 0, b_{k+1}, b_{k+2} \dots, b_d)^\top \in \mathcal{S}_2 && (26) \end{aligned}$$

we get

$$\begin{aligned} |\mathbf{A} - \mathbf{B}|^2 &= (a_1 - b_1)^2 + a_2^2 + \dots + a_k^2 + b_{k+1}^2 + \dots + b_d^2 = \\ &= a_1^2 + a_2^2 + \dots + a_k^2 + (c - b_1)^2 + b_{k+1}^2 + \dots + b_d^2 - \\ &\qquad\qquad\qquad\qquad - c^2 + 2cb_1 - 2a_1 b_1 = \\ &= r_1^2 + r_2^2 - c^2 + 2cb_1 - 2a_1 b_1 = 2b_1(c - a_1). \end{aligned}$$

Now $(c - a_1)$ factors out in (21) which becomes $\sqrt{\tilde{b_1}} : \sqrt{\bar{b_1}}$ (independent on the point $\mathbf{A}$), while b_1 factors out in (22) which becomes $\sqrt{c - \tilde{a_1}} : \sqrt{c - \bar{a_1}}$ (independently of the point $\mathbf{B}$). $\square$

Remark 18 It is natural to extend the notion of CL-related spheres to two degenerate cases. Suppose that $\mathcal{S}_1$ is a usual sphere but the radius r_2 of the second sphere is 0. Then $\mathcal{S}_2$ degenerates into a point of $\mathcal{S}_1$. Similarly if $r_2 \to \infty$, the sphere $\mathcal{S}_2$ tends to the $(\ell - 1)$-dimensional Euclidean subspace totally orthogonal to $\mathbb{E}''_\ell$ and passing through the center $\mathbf{C}_1$. In the remainder of the paper, we will include these two limit positions of the "sphere" $\mathcal{S}_2$ in the definition of CL-related spheres.

With this extension we can formulate two observations which follow directly from the definition of CL-related spheres.

Lemma 19 *Let* $\mathcal{S}_1$ *be a sphere of dimension* $(k - 1)$*. Then all* $(\ell - 1)$*-dimensional spheres which are CL-related to* $\mathcal{S}_1$

(1) fill the space $\mathbb{E}_d$*, i.e., through any point of the space there is precisely one of them;*
(2) are perpendicular to any k*-dimensional sphere containing* $\mathcal{S}_1$*.*

Example 20 Consider the case $d = 3$, $k = 2$. Then two CL-related spheres are just two usual circles. Fig. 1 shows one green circle and three orange circles which are chord-length related with it. By Remark 18, also the green axis is chord-length related with the green circle, as is any point on this circle. All orange circles are perpendicular to the spheres containing the green circle.

Proposition 21 *Let points $\mathbf{A}_0, \ldots, \mathbf{A}_k$ be points in general position in $\mathbb{E}_d$, $1 < k \leq n$, and $r_0, \ldots, r_k$ a collection of real non-negative numbers. Then the set of points $\mathbf{X} \in \mathbb{E}_d$ satisfying for all i, j*

$$|\mathbf{X} - \mathbf{A}_i| : |\mathbf{X} - \mathbf{A}_j| = r_i : r_j \tag{27}$$

is either empty or it is a sphere which is CL-related with the $(k-1)$-dimensional circumsphere of the points $\mathbf{A}_0, \ldots, \mathbf{A}_k$.

Proof. If more than one r_i is 0 then there is no such a point $\mathbf{X}$ since only one of $|\mathbf{X} - \mathbf{A}_i|$ can be 0. Without loss of generality we suppose $r_0 \neq 0$ and restrict ourselves to $j = 0$ in (27) since all other equations follow. Each of the k equations for $i = 1, \ldots, k$ defines a $(d-1)$-dimensional sphere (Apollonius' definition) degenerating to a hyperplane if $r_i = r_0$ and into the point $\mathbf{A}_i$ if $r_i = 0$. The centers of these spheres are in general position and therefore their intersection is either empty or it is a $(d-k)$-dimensional sphere (possibly degenerating in a plane if all r_i are equal or into a point of the circumsphere - see Remark 18). □

Definition 22 *Let $\mathcal{S} \subset \mathbb{E}_k$ be a $(k-1)$-dimensional sphere and $\gamma(\mathbf{u})$, where $\mathbf{u} = (u_1, \ldots, u_k)$, a rational parameterization of the space $\mathbb{E}_k$. We say that a k-dimensional parametric variety $\mathbf{P}(\mathbf{u})$ is RCL with respect to $\mathcal{S}$ and γ if*

$$|\mathbf{P}(\mathbf{u}) - \tilde{\mathbf{A}}| : |\mathbf{P}(\mathbf{u}) - \bar{\mathbf{A}}| = |\gamma(\mathbf{u}) - \tilde{\mathbf{A}}| : |\gamma(\mathbf{u}) - \bar{\mathbf{A}}| \tag{28}$$

holds for any $\tilde{\mathbf{A}}, \bar{\mathbf{A}} \in \mathcal{S}$ and any $\mathbf{u}$.

Remark 23 In particular examples γ will be a simple linear parameterization of the subspace or it will even coincide with its coordinates.

Since the notion of being RCL is clearly invariant with respect to similarities, we choose the reference sphere $\mathcal{S}$ as the unit sphere in the $(x_1, \ldots, x_k)$-plane, which we will call the *reference sphere*.

Let M be the inversion with respect to the $(d-1)$-dimensional sphere centered at

$$(0, \ldots, 0, \underbrace{-1}_{k-\text{th}}, 0, \ldots, 0)^\top$$

with radius $\sqrt{2}$. The inversion M maps the reference sphere to the $(x_1, \ldots, x_{k-1})$-space and all k-dimensional spheres containing the reference sphere are mapped to all k-dimensional spaces containing the $(x_1, \ldots, x_{k-1})$-space.

Since the inversion preserves circular objects and is conformal, any sphere which is CL-related to $\mathcal{S}$ is mapped by M to a sphere which has its center on the $(x_1, \ldots, x_{k-1})$-space and it spans a $(d-k+1)$-dimensional space whichis perpendicular to the $(x_1, \ldots, x_{k-1})$-space. These spheres fill the Euclidean space $\mathbb{E}_d$.

Let $\gamma(\mathbf{u})$ be given and $\mathbf{P}(\mathbf{u})$ be an RCL variety with respect to $\mathcal{S}$ and γ. Then for any $\mathbf{u}$ the two points $\gamma(\mathbf{u})$, $\mathbf{P}(\mathbf{u})$ lie on the same sphere which is CL-related to $\mathcal{S}$. The images $M(\gamma(\mathbf{u}))$, $M(\mathbf{P}(\mathbf{u}))$ will therefore lie on a sphere described in the previous paragraph, i.e., if

$$M(\gamma(\mathbf{u})) = (\tilde{x}_1(\mathbf{u}), \ldots, \tilde{x}_{k-1}(\mathbf{u}), \tilde{x}_k(\mathbf{u}), 0, \ldots, 0)^\top \tag{29}$$

and

$$M(\mathbf{P}(\mathbf{u})) = (\tilde{x}_1(\mathbf{u}), \ldots, \tilde{x}_{k-1}(\mathbf{u}), \bar{x}_k(\mathbf{u}), \ldots, \bar{x}_n(\mathbf{u}))^\top, \tag{30}$$

where

$$\bar{x}_k(\mathbf{u})^2 + \ldots + \bar{x}_n(\mathbf{u})^2 = \tilde{x}_k(\mathbf{u})^2.$$

Consequently,

$$\frac{(\bar{x}_k(\mathbf{u}), \ldots, \bar{x}_n(\mathbf{u}))^\top}{\tilde{x}_k(\mathbf{u})} \tag{31}$$

is a rational mapping on the unit $(d-k)$-dimensional sphere. However, all such mappings can be obtained via the stereographic projection which has the form

$$N(q_1, \ldots, q_{d-k}) = \frac{(1 - q_1^2 - \ldots - q_{d-k}^2, 2q_1, \ldots, 2q_{d-k})^\top}{1 + q_1^2 + \ldots + q_{d-k}^2}. \tag{32}$$

Noting that $M = M^{-1}$ we obtain the following result which is a generalization of (18):

Theorem 24 *A variety $\mathbf{P}(\mathbf{u})$ is a rational chord length parameterization with respect to the $(k-1)$-dimensional reference sphere $\mathcal{S}$ and the rational parameterization $\gamma(\mathbf{u})$ if and only if there exists a rational mapping $\mathbf{q}(\mathbf{u}) : \mathbb{R}^k \to \mathbb{R}^{d-k}$ such that*

$$\mathbf{P}(\mathbf{u}) = M\Big(\tilde{x}_1(\mathbf{u}), \ldots, \tilde{x}_{k-1}(\mathbf{u}), \tilde{x}_k(\mathbf{u}) N(\mathbf{q}(\mathbf{u}))\Big), \tag{33}$$

where $\tilde{x}_i$ is defined by (29).

5.2 Special cases

We show that the earlier results can be obtained as special cases of the general theory.

The case $d = 2$, $k = 1$. We obtain RCL curves in the plane. The reference sphere consists of two points $(\pm 1, 0)^\top$ and the inversion is

$$M(x,y)=\frac{1}{(x+1)^2+y^2}\begin{pmatrix}1-x^2-y^2\\2y\end{pmatrix}. \tag{34}$$

Let us consider the trivial parameterization of the x-axis $\gamma(u)=(u,0)^\top$. Then

$$M(\gamma(u))=\left(\frac{1-u^2}{(u+1)^2},0\right)^\top.$$

Moreover, N depends only on one rational function q and takes the form

$$N=\left(\frac{1-q^2}{1+q^2},\frac{2q}{1+q^2}\right)^\top.$$

Substituting these explicit formulas to (33) we obtain the following explicit formula, which is equivalent to the result obtained in Sánchez-Reyes and Fernández-Jambrina (2008).

Theorem 25 *A planar curve* $\mathbf{P}$ *is a rational chord length parameterization with respect to the reference points* $(\pm 1,0)^\top$ *if and only if there exists a rational function* $q(u)$ *such that*

$$\mathbf{P}(u)=\left(\frac{(1+q^2)u}{1+q^2u^2},\frac{q(1-u^2)}{1+q^2u^2}\right)^\top. \tag{35}$$

The case $d=3$, $k=2$. Using the inversion (17) and N depending only on one rational function q

$$N=\left(\frac{2q}{1+q^2},\frac{1-q^2}{1+q^2}\right)^\top,$$

we obtain the formula (10) for the standard parameterization of the plane $\gamma(u,v)=(u,v)^\top$.

The case $d=3$, $k=1$. In this case, the stereographic projection takes the form

$$M(x,y,z)=\frac{1}{(x+1)^2+y^2+z^2}\begin{pmatrix}1-x^2-y^2-z^2\\2y\\2z\end{pmatrix} \tag{36}$$

and N depends on two rational functions q_1,q_2

$$N=\left(\frac{1-q_1^2-q_2^2}{1+q_1^2+q_2^2},\frac{2q_1}{1+q_1^2+q_2^2},\frac{2q_2}{1+q_1^2+q_2^2}\right)^\top.$$

We obtain the following explicit formula for the standard parameterization of the x-axis $\gamma(u) = (u, 0, 0)^\top$.

Theorem 26 *A curve* $\mathbf{P}$ *is a rational chord length parameterization with respect to the reference points* $(0, \pm 1)^\top$ *if and only if there exist rational functions* $q_1(u)$, $q_2(u)$ *such that*

$$\mathbf{P}(u) = \left(\frac{(1 + q_1^2 + q_2^2)u}{1 + u^2(q_1^2 + q_2^2)}, \frac{(1 - u^2)q_1}{1 + u^2(q_1^2 + q_2^2)}, \frac{(1 - u^2)q_2}{1 + u^2(q_1^2 + q_2^2)} \right)^\top. \qquad (37)$$

Example 27 In particular, if we set $q_2(u) = kq_1(u)$, $k \in \mathbb{R}$, then we obtain curves with vanishing torsion, i.e., all associated RCL curves are planar. Especially, if both functions are constant we obtain for the curvature and the torsion

$$\kappa = \frac{2\sqrt{q_1^2 + q_2^2}}{1 + q_1^2 + q_2^2} = const., \qquad \tau = 0, \qquad (38)$$

i.e., these curves are circles.

6 Conclusion

We described a class of rational triangular and quadrangular Bézier surfaces possessing a parameterization which preserves the distance ratios to the vertices of the domain triangle or rectangle inscribed to the reference circle. This extends the property of chord-length parameterization of rational curves, which was studied in Lü (2009) and Sánchez-Reyes and Fernández-Jambrina (2008), to the case of surfaces. We identified a family of RCL surfaces, characterized their general parameterization and studied their properties. We have also presented a general pattern for RCL varieties of any dimension. This general result allows to express all planar curves, spatial curves and surfaces with RCL property by one general formula. The future research will be focused mainly on modelling with surface patches of this type.

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