Local and adaptive refinement with hierarchical B–splines*

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Abstract. – *Adaptive spline models for geometric modeling and spline-based PDEs solvers have recently attracted increasing attention both in the context of computer aided geometric design and isogeometric analysis. In particular, approximation spaces defined over extensions of tensor-product meshes which allow axis aligned segments with T–junctions are currently receiving particular attention. In this short paper we review some recent results concerning the characterization of the space spanned by the hierarchical B–spline basis. In addition, we formulate a refinement algorithm which allows us to satisfy the conditions needed for this characterization.*

1. – Introduction

By using the same smooth function spaces for describing the geometry and performing the simulation phase, the recently introduced isogeometric approach [8] will potentially lead to major improvements of the product design process in certain parts of industrial applications. The geometry of the computational domain is usually defined in terms of non–uniform rational B–splines (NURBS), the standard representation model used in geometric modeling. Classic finite element methodologies transform the NURBS model provided by commercial computer aided design (CAD) systems into an *approximate* triangular or polygonal geometry described by piecewise linear or non–linear elements. On the other hand, the goal of the isogeometric approach is to preserve the *exact* CAD geometry by eliminating the time–consuming mesh generation task and avoiding additional interactions with the CAD system during the mesh refinement procedure.

In order to extend the isogeometric methodology with spline representations which allow local control of the refinement procedure, suitable applications of the T–spline model and related issues have also been considered [1, 2]. Recently, alternative solutions based on hierarchical B–splines have been investigated [11].

Starting from the classical construction of hierarchical tensor-product B–splines, we will review some recent results concerning the characterization of multilevel spline spaces and the identification of suitable hierarchical bases. We will also present a simple hierarchical refinement strategy which allows to guarantee certain properties of the corresponding multilevel spline space.

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2. – Spline hierarchy

Hierarchical B–splines constitutes an effective extension of classical tensor-product B–splines to address the problem of a local and adaptive mesh refinement in standard approximation algorithms [3, 4]. A sequence of nested tensor–product meshes is used to identify different levels of resolution. B–splines defined with respect to these increased levels of details are then suitably used over the regions selected to be refined at the current refinement step.

We consider a sequence of tensor–product grids which corresponds to a nested sequence of bivariate B–spline spline spaces $V^0, \ldots, V^{N-1}$ so that $V^{\ell-1} \subset V^\ell$, for $\ell = 1, \ldots, N-1$. At each level ℓ, the bi–degree and smoothness are (d, d) and $(d-1, d-1)$, respectively. The initial spline space V^0 is spanned by a bivariate tensor–product B–spline basis $\mathcal{B}^0$ with respect to two bi–infinite uniform knot sequences. Any space V^ℓ instead is spanned by the normalized B–spline basis $\mathcal{B}^\ell$ defined on corresponding nested knot sequences obtained by subsequently applying dyadic subdivision to the previous level.

Let $\{\Omega^\ell\}_{\ell=0,\ldots,N-1}$ be a nested sequence of subdomains so that $\Omega^{\ell-1} \supseteq \Omega^\ell$, for $\ell = 1, \ldots, N-1$. For each hierarchical level, Ω^ℓ is defined as a collection of cells with respect to the tensor–product grid of level $\ell - 1$. We say that a B–spline $\beta \in \mathcal{B}^\ell$ is *active* if and only if its support is completely contained in Ω^ℓ but not in $\Omega^{\ell+1}$, and *passive* otherwise. A simple selection mechanism from a starting sequence of B-spline bases allows to construct a basis for the corresponding multilevel spline space [9, 10, 11].

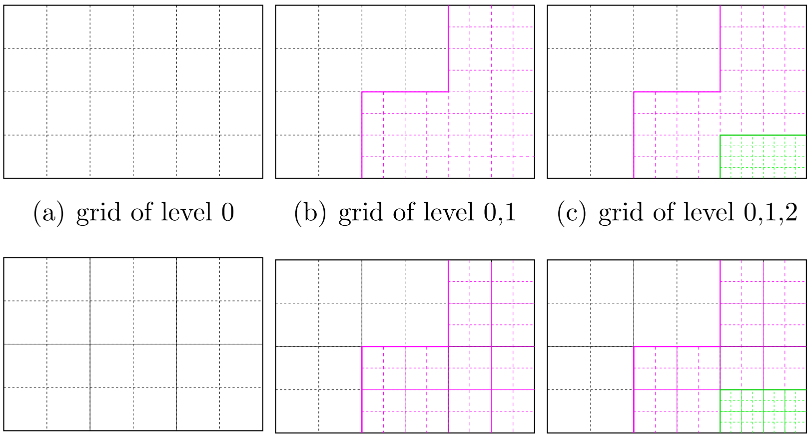
Definition 2.1. *The hierarchical B–spline basis is defined as the set of all* active *B–splines over the tensor–product grid of each level.*

The application of the hierarchical B–spline construction to scattered data approximations and to the solution of partial differential equations in isogeometric analysis has already shown the potential of this refinement framework with respect to the modeling of detailed local features and to the reduced number of degrees of freedom needed to obtain a certain accuracy, see e.g., the recent work [11].

3. – Hierarchical spline spaces

Related issues to address naturally concern the characterization of the hierarchical spline space. In particular, we may investigate the possibility of representing any function of arbitrary multi-degree and with maximum order of smoothness. By focusing on the bivariate case, it is possible to show that, under some reasonable assumptions on the domain configuration, we can give a positive answer to this question.

Let S^ℓ be the the space of piecewise polynomials of bi–degree (d, d) on the subdivision of the plane obtained by restricting the grid of V^ℓ to $\Omega^0 \setminus \Omega^{\ell+1}$ for a certain level ℓ. In the considered uniform setting characterized by single knots at all levels, in order to prove that the number of B–splines whose support overlap $\Omega^0 \setminus \Omega^{\ell+1}$ is equal to the dimension of S^ℓ, we need the following assumption.



(a) grid of level 0 (b) grid of level 0,1 (c) grid of level 0,1,2

(d) 2–grid of level 0 (e) 2–grid of level 0,1 (f) 2–grid of level 0,1,2

Figure 1: Grids and d–grids for $d = 2$.

(A) Any pair of B–splines of level ℓ whose supports intersect opposite sides along the boundary of $\Omega^0 \setminus \Omega^{\ell+1}$ do not overlap each other.

The complete analysis leading to the above assumption is carried over in [7] by introducing the notion of *offset* at a certain distance to a given domain and leads to the following result.

Theorem 3.1. *[7] If Assumption (A) on the domain configuration holds for all $\ell = 0, \ldots, N-1$, then the hierarchical B–spline basis introduced in Definition 2.1 spans the entire space W defined as*

$$W = \left\{ f \in \mathcal{C}^{d-1,d-1}(\Omega^0) : \left. f \right|_{\Omega^0 \setminus \Omega^{\ell+1}} \in S^\ell \; \forall \ell = 0, \ldots, N-1 \right\}.$$

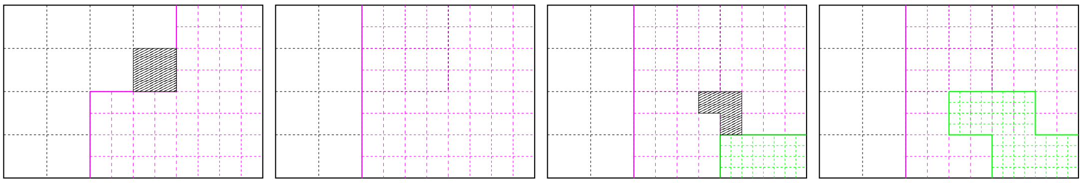
Hence, the space spanned by hierarchical B–splines contains all spline functions of a certain degree and maximum smoothness that exists on the underlying hierarchical grids.

4. – A simple refinement algorithm

Finally, we present a method which allows to perform adaptive refinement while satisfying the condition which was formulated in the previous section. We assume that, for any level, the number of cells in each direction, is a multiple of d. By denoting as *d–grid* of level ℓ the *aligned disjoint* boxes composed of $d \times d$ cells with respect to the grid of V^ℓ (see Figure 1), an algorithm to guarantee satisfaction of assumption (A) is based on the following observation, see also Remark 21 in [7].

Remark 1. *If $\Omega^{\ell+1}$, for $\ell = 0, \ldots, N-2$, can be decomposed into a d–grid of level ℓ, then assumption (A) is satisfied.*

A simple refinement procedure based on the above observations may be summarized as follows.



(a) $\Phi \cap \Omega^0$ (hatched) (b) identification of $\hat{\Omega}^1$ (c) $\Phi \cap \Omega^1$ (hatched) (d) identification of $\hat{\Omega}^2$

Figure 2: Identification of $\hat{\Omega}^1$ and $\hat{\Omega}^2$ in Example 1.

Algorithm 1.

Input:

I1 A nested hierarchy of domains $\{\Omega^\ell\}_{\ell=0,\ldots,N}$ so that $\Omega_0 \supseteq \Omega_1 \supseteq \ldots \supseteq \Omega_N$, where each Ω^ℓ can be decomposed into a d–grid of level ℓ and $\Omega^N = \emptyset$;

I2 A set Φ of cells marked to be refined.

1. $\hat{\Omega}_0 = \Omega_0$;
2. for $\ell = 0, \ldots, N-1$

$$\hat{\Omega}^{\ell+1} = \Omega^{\ell+1} \cup C$$

where C is the union of all the cells c which belong to the d–grid of level ℓ so that $c \cap \Phi \neq \emptyset$;

Output: the enlarged hierarchy of domains $\{\hat{\Omega}^\ell\}_{\ell=0,\ldots,N}$ so that $\hat{\Omega}_0 \supseteq \hat{\Omega}_1 \supseteq \ldots \supseteq \hat{\Omega}_N$, where each $\hat{\Omega}^\ell \supseteq \Omega^\ell$ is the union of *disjoint* boxes composed by $d \times d$ cells with respect to the grid of $V^{\ell-1}$.

Example 1. By considering the input data illustrated in Figure 2(a) and (c) with $d = 2$, the application of step 2 of Algorithm 1 allows to identify $\hat{\Omega}^1$ and $\hat{\Omega}^2$ as shown in Figure 2(b) and (d).

5. – Closure

Bases and dimensions of bivariate hierarchical tensor–product splines were recently investigated [7] together with specific condition on admissible domain configurations. In this paper we presented a simple algorithm which ensures the repeated fulfillment of this condition during the refinement procedure.

The hierarchical framework can even be adapted to the multivariate setting or to more general spline spaces [5, 6]. The possibility of extending the analysis of hierarchical spline space and the proposed refinement algorithm to a more general multivariate setting is an interesting topic for future research.

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