Voronoi Diagrams from (Possibly Discontinuous) Embeddings

Mario Kapl
Insitute of Applied Geometry
Johannes Kepler University Linz
4040 Linz, Austria
Email: mario.kapl@jku.at

Franz Aurenhammer
Institute for Theoretical Computer Science
University of Technology Graz
8010 Graz, Austria
Email: auren@igi.tu-graz.ac.at

Bert Jüttler
Insitute of Applied Geometry
Johannes Kepler University Linz
4040 Linz, Austria
Email: bert.juettler@jku.at

***Abstract*—We introduce a new metric framework which is based on an injective embedding of $[0,1]^2$ into $\mathbb{R}^m$, for $m \geq 2$, and an additional scaling function for re-scaling the distances. The framework is used to construct a new type of generalized Voronoi diagrams in $[0,1]^2$, which is possibly anisotropic. We present different possible applications of these Voronoi diagrams with several examples of generated diagrams.**

***Keywords*-Voronoi diagram; embedding; distance graph;**

I. INTRODUCTION

The Voronoi diagram is a well known concept in geometry with a wide range of possible applications, e.g. motion planing, geometric clustering and meshing, cf. [1]. Beside the classical Euclidean Voronoi diagram there exists a large number of generalizations of this structure.

In this paper we generalize it by introducing a new metric framework, referred to as scaled embedding generated (SEG) metric, which is based on a one-to-one embedding of $[0,1]^2$ into $\mathbb{R}^m$, for $m \geq 2$, and an additional scaling function for re-scaling the distances. The topic of this paper is an extension of our work in [2] in two directions. On the one hand we describe a generalization of the SEG metric to discontinuous embeddings. On the other hand we present several applications to Voronoi diagrams for demonstrating the power of this framework.

This paper is organized as follows. In Section II, we introduce the concept of the SEG metric and use it to define the so-called SEG metric Voronoi diagrams. This new class of Voronoi diagrams includes Voronoi diagrams which are possibly anisotropic, since the SEG metric reflects the anisotropy of the parameter lines of the associated embedding. The SEG metric Voronoi diagram offers several advantages compared to other types of anisotropic Voronoi diagrams, e.g. [3], [4] and [5]. In our approach the computation of the single distances is fast and simple and the used distance function really defines a metric. Moreover, the SEG metric Voronoi diagram can be generated by intersecting an Euclidean Voronoi diagram in $\mathbb{R}^m$ with the associated embedding.

Section III describes different possible applications of the SEG metric Voronoi diagrams, including diagrams for Poincaré metric-like distances, diagrams for distance discontinuities and diagrams from distance graphs. The power of this new class of Voronoi diagrams is presented with the help of several examples. In Section IV, we conclude this paper and describe some possible future work.

II. SEG METRIC VORONOI DIAGRAMS

We describe a new metric framework and use it to define a class of anisotropic Voronoi diagrams.

A. The SEG metric framework

Definition 1: Let $\mathbf{x} : [0,1]^2 \to \mathbb{R}^m$, for $m \geq 2$, be a one-to-one embedding with $\mathbf{x}(u,v) = (x_1(u,v), \ldots, x_m(u,v))$. In addition, let $r \mapsto d(r)$, for $r \geq 0$, be a scalar-valued scaling function with the following properties:

- $d(0) = 0$
- $d(r) > 0$, for $r > 0$
- $d'(r) \geq 0$, for $r \geq 0$
- $d(r)/r$ is monotonically decreasing, for $r > 0$.

The distance D between two points $\mathbf{u}_1 = (u_1, v_1)$ and $\mathbf{u}_2 = (u_2, v_2)$ in $[0,1]^2$ is given by

$$D(\mathbf{u}_1, \mathbf{u}_2) = d(||\mathbf{x}(u_1, v_1) - \mathbf{x}(u_2, v_2)||), \tag{1}$$

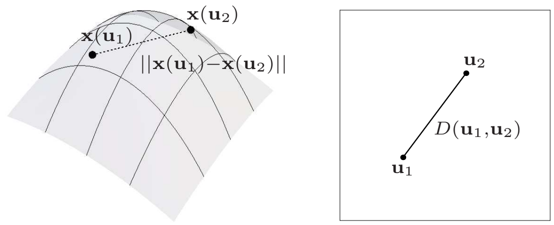
see Fig. 1.

Note that in contrast to [2], the embedding $\mathbf{x}(u,v)$ need not to be continuous. The distance D defines again a metric on $[0,1]^2$. The proof of this proposition works analogously as the one in [2]. We will call D the *scaled embedded generated (SEG) metric*. For $m = 2$ or $m = 3$, the embedding $\mathbf{x}(u,v)$ is a parametric surface without self-intersections in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

For simplicity we have chosen $[0,1]^2$ for the parameter domain of the embedding $\mathbf{x}(u,v)$. But in general, the domain can be extended to each subset of $\mathbb{R}^2$. Two possible scaling functions are

$$d(r) = ar \quad \text{and} \quad d(r) = a\ln(br+1)$$

for suitable constants $a, b > 0$. In order to simplify the paper we will restrict ourselves to $d(r) = r$ for the examples in Section III.



(a) (b)

Figure 1. (a) The concept of the scaled embedded generated (SEG) metric. The distance $||\mathbf{x}(\mathbf{u}_1) - \mathbf{x}(\mathbf{u}_2)||$ between the points $\mathbf{x}(\mathbf{u}_1)$ and $\mathbf{x}(\mathbf{u}_2)$ on the embedding $\mathbf{x}(u, v)$ (b) leads to the distance $D(\mathbf{u}_1, \mathbf{u}_2)$ between the points $\mathbf{u}_1$ and $\mathbf{u}_2$ in the parameter domain $[0, 1]^2$.

B. SEG metric Voronoi diagrams

We will use the SEG metric framework to generalize the concept of Voronoi diagrams in $[0, 1]^2$ in a canonical way.

Definition 2: Let $\mathbf{P} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$ be a finite set of points (called sites) in $[0, 1]^2$ and let D be a SEG metric, given by (1). We define the *Voronoi cell* of a site $\mathbf{u}_i \in \mathbf{P}$ with respect to D as the open set

$$V_D^i(\mathbf{P}) = \{\mathbf{u} \in [0,1]^2 \mid D(\mathbf{u}, \mathbf{u}_i) < D(\mathbf{u}, \mathbf{u}_j) \text{ for all } j \neq i\}.$$

The *Voronoi diagram* $V_D(\mathbf{P})$ with respect to D is given by the complement of all Voronoi cells in $[0, 1]^2$,

$$V_D(\mathbf{P}) = [0,1]^2 \setminus \left(\bigcup_i V_D^i(\mathbf{P}) \right).$$

We will call $V_D(\mathbf{P})$ the *SEG metric Voronoi diagram*.

The SEG metric Voronoi diagram can be generated with the help of an Euclidean Voronoi diagram in $\mathbb{R}^m$, see Fig. 2. The construction consists of the following steps:

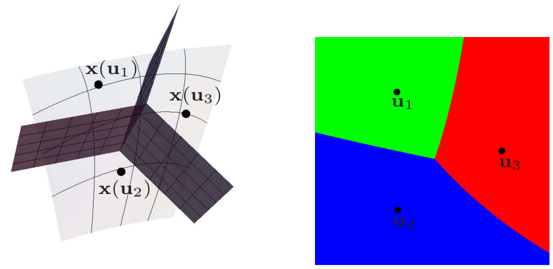
- Given the set of sites $\mathbf{P} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$ with $\mathbf{u}_i \in \mathbb{R}^2$, and the SEG metric D, which is defined by the embedding $\mathbf{x}(u, v)$. We calculate the points $\mathbf{x}_i = \mathbf{x}(\mathbf{u}_i)$ on the embedding $\mathbf{x}(u, v)$.
- For the obtained set of points $\mathbf{P}_\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ we compute the corresponding Euclidean Voronoi diagram in $\mathbb{R}^m$.
- We intersect the resulting Voronoi cells of this Euclidean Voronoi diagram with the embedding $\mathbf{x}(u, v)$, which leads to a Voronoi diagram on $\mathbf{x}(u, v)$. The corresponding parameter values $(u, v) \in [0, 1]^2$ of $\mathbf{x}(u, v)$ defines the desired SEG metric Voronoi diagram $V_D(\mathbf{P})$ in $[0, 1]^2$.

III. Applications

We present several applications of the SEG metric Voronoi diagrams. For all described examples below the scaling function is set to the identity, $d(r) = r$.

A. Poincaré metric-like distances[1]

Let $\mathbf{c} \in [0, 1]^2$ and let $t \mapsto r(t)$ for $t \geq 0$ be a scalar-valued function, which is assumed to have the following properties:



(a) (b)

Figure 2. (a) Construction of the SEG metric Voronoi diagram by intersecting the embedding $\mathbf{x}(u, v)$ with an Euclidean Voronoi diagram in $\mathbb{R}^m$; (b) The corresponding parameter values of the embedding defines the SEG metric Voronoi diagram in $[0, 1]^2$.

- $r(0) = 0$
- $r(t) > 0$, for $t > 0$
- $r'(t) > 0$, for $t \geq 0$.

We generate a SEG metric fulfilling

$$D(\mathbf{u}, \mathbf{c}) = r(||\mathbf{u} - \mathbf{c}||) \text{ for all } \mathbf{u} \in [0, 1]^2.$$

The associated embedding $\mathbf{x}(u, v)$ of the desired SEG metric can be constructed as follows:

$$\mathbf{x}(\mathbf{u}) = \begin{cases} (0,0), & \text{for } \mathbf{u} = \mathbf{c}, \\ (\mathbf{u} - \mathbf{c})\frac{r(||\mathbf{u}-\mathbf{c}||)}{||\mathbf{u}-\mathbf{c}||}, & \text{otherwise.} \end{cases}$$

In the following example we use this framework to construct a Poincaré metric-like distance function and generate some SEG metric Voronoi diagrams with the help of this metric.

Example 3: We consider the point $\mathbf{c} = (0.5, 0.5)$ and the function $r(t) = (2t)^3$. For this initialization we obtain a SEG metric for which some generalized circles with a uniform sequence of radii with center $\mathbf{c}$ and the associated embedding $\mathbf{x}(u, v)$ are visualized in Fig. 3(a) and (b), respectively.

In addition, Fig. 3 displays an example of a resulting SEG metric Voronoi diagram in (c), and the corresponding Euclidean Voronoi diagram in (d). The sites are chosen uniformly distributed, i.e as the vertices of a triangular mesh.

B. Modeling distance discontinuities

We consider the domain $[0, 1]^2$, which is decomposed into a finite number of disjoint subdomains $\Omega_0 \ldots, \Omega_n$, such that

$$\bigcup_{i=0}^{n} \Omega_i = [0, 1]^2.$$

An example of such a decomposition of the domain $[0, 1]^2$ into five disjoint subdomains is shown in Fig. 4(a).

We will define a piecewise embedding $\mathbf{x}(u, v)$ on the decomposition of the domain to describe the following situation. Assume that the domain $[0, 1]^2$ describes a part of a map, where different obstacles such as rivers, fences etc. are given. Crossing these obstacles costs resources, which is modeled by an increase of the distance.

Example 4: We consider the decomposition of the domain $[0, 1]^2$ into five subdomains shown in Fig 4, where

[1] This metric is inspired by the Poincaré metric on the unit disk which provides a model of the hyperbolic plane.

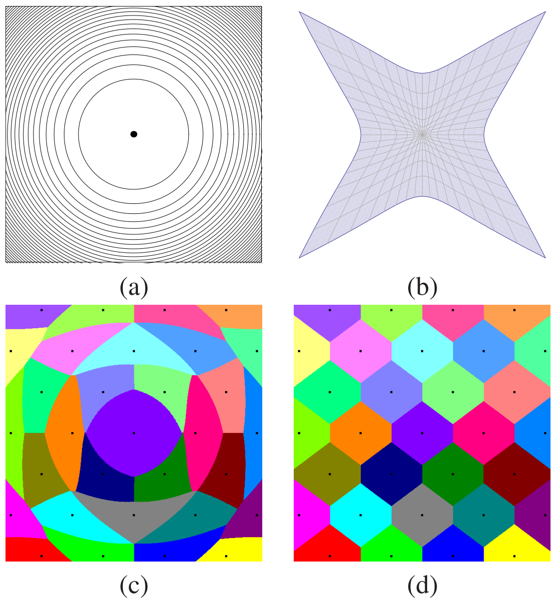


Figure 3. (a) Generalized circles for a uniform sequence of radii with the center $\mathbf{c} = (0.5, 0.5)$, described by $D(\mathbf{u}, \mathbf{c}) = (2||\mathbf{u} - \mathbf{c}||)^3$; (b) The associated embedding of $[0, 1]^2$ into $\mathbb{R}^2$; (c)-(d) Example of a SEG metric Voronoi diagram and the corresponding Euclidean Voronoi diagram.

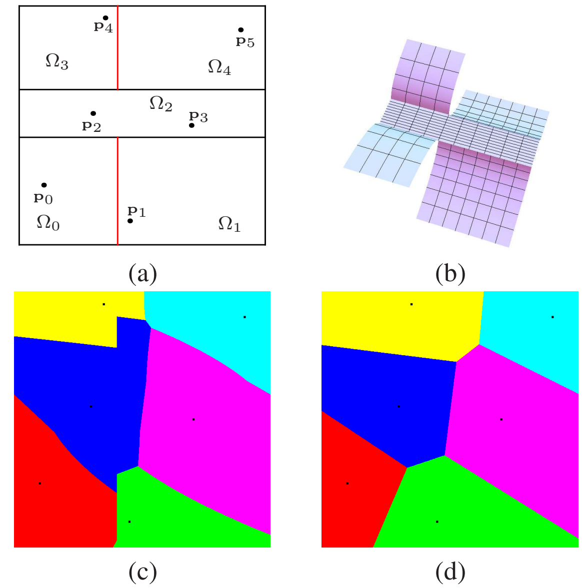


Figure 4. (a) Decomposition of the parameter domain $[0, 1]^2$; (b) The piecewise embedding, which induces the SEG metric for modeling the distance discontinuities; (c) The SEG metric Voronoi diagram and (d) the Euclidean Voronoi diagram with the supply stations $\mathbf{p}_i$ as sites.

the two red lines specify a river with a bridge. In addition, the points $\mathbf{p}_i$ describe six supply stations for gas.

The task is to build a pipeline from any point of the domain $[0, 1]^2$ to the nearest supply station $\mathbf{p}_i$ with respect to the side constraint, that crossing a river, except over the bridge, costs a certain fixed amount of resources. We also make the simplifying assumption that the pipeline needs to be straight. We define now a SEG metric and a Voronoi diagram with the supply stations as sites, which reflects this situation.

Depending on the height of the cost for the crossing, we can generate a piecewise embedding $\mathbf{x}(u, v)$, which is visualized in Fig. 4(b). The obtained SEG metric Voronoi diagram and the corresponding Euclidean Voronoi diagram with the supply stations $\mathbf{p}_i$ as sites are presented in Fig 4(c) and (d), respectively.

Due to the desired continuity of the embedding between Ω_2 and the neighboring subdomains, the cost for crossing the obstacle goes to zero when getting closer to the bridge. But this is also quite realistic in application, since the pipeline might then make a detour via the bridge, thereby avoiding of crossing the obstacle. Also, the effect of the obstacle decreases for larger distances to it. Again this can be justified by the possibility of using a detour.

The SEG metric Voronoi diagrams shows clearly the influence of the metric discontinuities.

C. Voronoi diagrams from distance graphs

In [2], we described a method to generate a SEG metric, i.e. the associated embedding $\mathbf{x}(u, v)$, which approximates a given distance graph G on n points $\mathbf{p}_i$ in $[0, 1]^2$. We required G to satisfy the generalized triangle inequality, that is, for each edge $(\mathbf{p}_\alpha, \mathbf{p}_\beta)$ the associated length $L_{\alpha,\beta}$ is at most the length of any existing path in G from $\mathbf{p}_\alpha$ to $\mathbf{p}_\beta$.

As a possible application of such a framework, we could generate a SEG metric, which gives us the travel times by car between cities and use this metric to construct a SEG metric Voronoi diagram.

Example 5: We consider the map and the distance graph from Fig. 5, which show 20 cities in Upper Austria, which is one of the nine provinces in Austria. The red streets on the map specify highways, the yellow ones are state roads. But beside these fast roads, there exist a lot of country roads connecting the neighboring cities.

The distance graph depicts some selected travel times by car between these cities. In Table I, we have enumerated the names of the cities, which represent the points $\mathbf{p}_i$ on the map and in the distance graph. The chosen parameter values $(u, v) \in [0, 1]^2$ for each point $\mathbf{p}_i$ correspond to the parameter values of the map in Fig. 5.

In Table II, we have listed the lengths $L_{\alpha,\beta}$ of the edges $(\mathbf{p}_\alpha, \mathbf{p}_\beta)$ in the distance graph given in Fig 5, which correspond to the expected travel times by car between the cities $\mathbf{p}_\alpha$ and $\mathbf{p}_\beta$ calculated by a navigation system.

We have used our graph fiting procedure from [2] for generating a cubic spline embedding $\mathbf{x}(u, v)$ into $\mathbb{R}^7$ to obtain a SEG metric with an accurate approximation. In Fig. 5, we have visualized the SEG metric Voronoi diagrams obtained by two different sets of sites. In (c), the sites are chosen uniformly distributed, in detail, as the vertices of a triangular mesh. For the result in (d), we have chosen for the sites the points $\mathbf{p}_i$ from the distance graph. In addition,

Table I
THE NAMES OF THE CITIES WHICH REPRESENT A POINT $\mathbf{p}_i$ ON THE MAP AND IN THE DISTANCE GRAPH GIVEN IN FIG. 5.

$\mathbf{p}_0$	Wels	$\mathbf{p}_7$	Gallneukirchen	$\mathbf{p}_{14}$	Ottensheim
$\mathbf{p}_1$	Eferding	$\mathbf{p}_8$	Enns	$\mathbf{p}_{15}$	Gramastetten
$\mathbf{p}_2$	St. Martin i. M.	$\mathbf{p}_9$	Steinhaus b. W.	$\mathbf{p}_{16}$	St. Veit i. M.
$\mathbf{p}_3$	Marchtrenk	$\mathbf{p}_{10}$	Scharten	$\mathbf{p}_{17}$	Ernsthofen
$\mathbf{p}_4$	Ansfelden	$\mathbf{p}_{11}$	Kleinzell i. M.	$\mathbf{p}_{18}$	Ried i. R.
$\mathbf{p}_5$	Linz	$\mathbf{p}_{12}$	Neuhofen a. K.	$\mathbf{p}_{19}$	Neumarkt i. M.
$\mathbf{p}_6$	Hellmonsödt	$\mathbf{p}_{13}$	Pasching		

Table II
THE EXPECTED TRAVEL TIMES BY CAR $L_{\alpha,\beta}$ BETWEEN TWO CITIES $\mathbf{p}_\alpha$ AND $\mathbf{p}_\beta$ IN THE DISTANCE GRAPH GIVEN IN FIG. 5.

$L_{0,3}$	9	$L_{3,4}$	15	$L_{5,7}$	13	$L_{8,17}$	15
$L_{0,9}$	11	$L_{3,9}$	17	$L_{5,8}$	23	$L_{8,18}$	15
$L_{0,10}$	16	$L_{3,10}$	20	$L_{5,13}$	16	$L_{9,12}$	20
$L_{1,2}$	18	$L_{3,12}$	15	$L_{5,14}$	12	$L_{11,14}$	24
$L_{1,10}$	11	$L_{3,13}$	15	$L_{5,15}$	18	$L_{11,16}$	26
$L_{1,11}$	28	$L_{5,5}$	16	$L_{5,19}$	22	$L_{12,17}$	33
$L_{1,13}$	18	$L_{4,8}$	15	$L_{6,15}$	20	$L_{14,15}$	13
$L_{1,14}$	21	$L_{4,12}$	15	$L_{6,16}$	22	$L_{15,16}$	16
$L_{2,11}$	13	$L_{4,13}$	15	$L_{6,19}$	28	$L_{17,18}$	25
$L_{2,14}$	17	$L_{4,17}$	27	$L_{7,18}$	20	$L_{18,19}$	24
$L_{2,16}$	21	$L_{5,6}$	21	$L_{7,19}$	15		

we have visualized the corresponding Euclidean Voronoi diagram in (e).

One may observe that the Voronoi cells with respect to the SEG metric take the highway layout and the geography (e.g. the location of rivers) into account.

IV. CONCLUSION

This paper has been an extension of our work in [2]. We have presented several applications of the SEG metric Voronoi diagram, which is possibly anisotropic and is a generalization of the Euclidean Voronoi diagram in $[0,1]^2$ in a canonical way. E.g., the SEG metric framework allows us to model distance discontinuities or to describe the travel times by car between cities.

As a possible topic for future research one might study computational methods for creating discontinuous embeddings modeling distances in the presence of obstacles.

For the case of a SEG metric induced by a smooth embedding, we described in [2] some conditions under which the resulting SEG metric Voronoi diagram is orphan-free. Similar conditions for Voronoi diagrams obtained by discontinuous embeddings could be of interest, too.

As a further topic of possible future work we can use the SEG metric framework to generate generalized medial axis for shapes.

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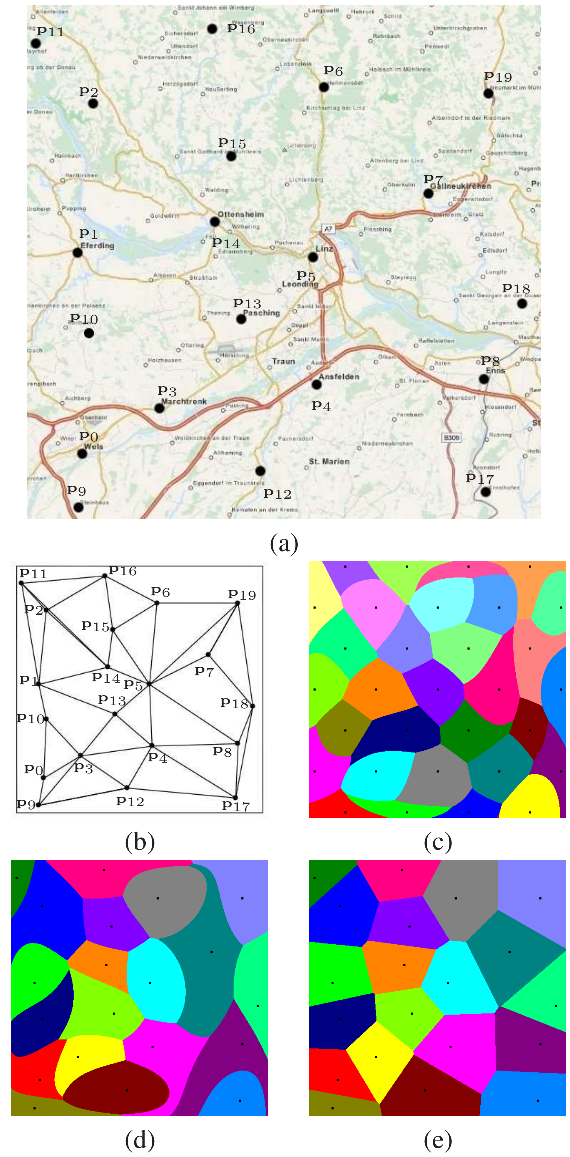

Figure 5. (a) The map[2] with the chosen cities $\mathbf{p}_i$ (see Table I); (b) The distance graph, which describes some selected times of travel by car $L_{\alpha,\beta}$ between the cities $\mathbf{p}_\alpha$ and $\mathbf{p}_\beta$ (see Table II); (c) The SEG metric Voronoi diagram for a set of uniformly distributed sites; (d) The SEG metric Voronoi diagram and (e) the Euclidean Voronoi diagram by choosing the points $\mathbf{p}_i$ for the sites.

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[2]The map was taken from *www.openstreetmap.org*.