Layered Reeb Graphs of a Spatial Domain*

B. Strodthoff †, B. Jüttler†

Abstract

We introduce layered Reeb graphs as a representation for the topological structure of Reeb spaces and sketch a boundary-based algorithm for computing them.

1 Introduction

Reeb graphs are topological graphs originating in Morse theory, which represent the topological structure of a Riemannian manifold based on a scalar-valued, sufficiently smooth function defined on it (see [1] for an introduction). The use of more than one function leads to Reeb spaces, which are thus able to capture more features of an object. Reeb spaces were considered in 2008 [4], but appear to be little researched by now, especially Reeb spaces for manifolds with boundary. In the first part of this work, we introduce the layered Reeb graph as a discrete representation for Reeb spaces of 3-manifolds with respect to two scalar-valued functions. After that we present a restricted class of defining functions, for which the layered Reeb graph can be computed from a boundary representation of the spatial domain of interest. This leads to substantial computational advantages if the manifold is given in a boundary description, since no volumetric description has to be constructed.

2 Definition of the layered Reeb graph

After defining Reeb graphs and -spaces, we present an approach for computing Reeb graphs or -spaces by a sweep algorithm. This motivates the definition of the layered Reeb graph.

2.1 Reeb graphs and -spaces

Consider n scalar-valued functions $f_1, \ldots, f_n$ defined on a d-manifold (or manifold with boundary) M. Any subset of these functions defines *level sets*, where all these functions have constant values. Connected parts of M on the same level set are called *level set components.*

Definition 1 *Points where level set components of all functions $f_1, \ldots, f_n$ or components of their boundary meet or disappear are called* critical, *all other points are regular. Contracting every level set component to a point gives the* Reeb space *of M with respect to $f_1, \ldots, f_n$. In the special case of $n = 1$ we speak of a* Reeb graph.

In the following, we will only consider three special cases of this definition:

- Reeb graph of a 2-manifold (Figures 1 and 2b),
- Reeb graph of a 3-manifold (Figures 2c-d), and
- Reeb space of a 3-manifold with respect to two functions (Figure 3).

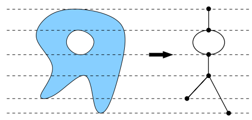

Figure 1: Reeb graph of a 2-manifold with boundary in the plane with respect to the height function, which maps each point to its last Cartesian coordinate.

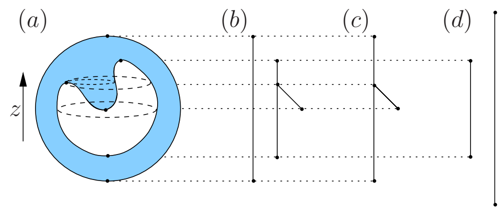

Figure 2: Reeb graphs of manifolds in space with respect to the height function. (a) The object: sphere with a bowl-shaped void. (b) Reeb graph of the surface considered as 2-manifold in space. (c) Reeb graph of the object. (d) Reeb graph of the air-volume surrounding the object.

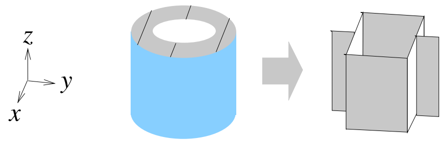

Figure 3: Reeb space of a vertical pipe with respect to the z and y- coordinate. Middle: Lines in the horizontal cut show common level sets. Right: Structure of the resulting Reeb space.

*Supported by ESF Programme EuroGIGA-Voronoi
†Johannes Kepler University, Linz, Austria

2.2 Sweep algorithm for computing Reeb spaces

In the remainder of this paper, we consider two sufficiently smooth, scalar-valued functions f and g on $\mathbb{R}^3$, and a 3-manifold M with boundary embedded in $\mathbb{R}^3$. We are interested in the Reeb space with respect to (f_1, f_2) with $f_1 = f|_M$ and $f_2 = g|_M$.

Consider a level set of f at function value $f = c$, and the restriction g_c of g onto this level set. The Reeb graph of the level set $f = c$ (considered as a 2-manifold with boundary) with respect to g_c contracts all those points of the level set with the same g_c-value. We denote these Reeb graphs as *level set Reeb graphs*.

A point of a level set Reeb graph thus represents a connected component of points of M which are mapped to the same value both by f and g. Sweeping through the level sets of f and continually connecting subsequent level set Reeb graphs gives the Reeb space of M with respect to (f, g).

This motivates the following *algorithm* for the computation of Reeb spaces: First, identify those points of M where changes occur in the structure of level sets of f or of their level set Reeb graphs. We will call these points *events* in the following. Then, sweep through the level sets of f. At each event, find the level set components in which changes occur, and update their level set Reeb graphs. These can be computed by a sweep inside the level set.

2.3 The layered Reeb graph

Consider the Reeb graph of M with respect to f. An arc of this graph represents an evolving level set component of f. This component's level set Reeb graph goes through structural changes only at certain event points. In the layered Reeb graph, the original Reeb graph arc is divided at these events, and each part stores its level set Reeb graph, see Figure 4.

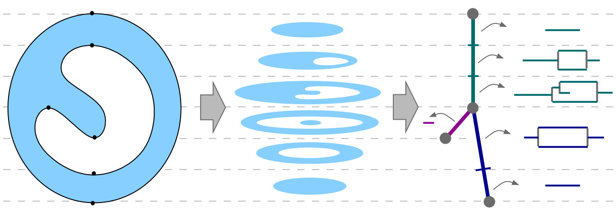

Figure 4: Layered Reeb graph. Left: vertical cut through a 3-manifold. Middle: level sets of f. Right: layered Reeb graph.

Definition 2 *The layered Reeb graph of a 3-manifold M with boundary with respect to two sufficiently smooth, scalar-valued functions f and g is obtained as follows. Take the Reeb graph with respect to the first function f and subdivide each arc into parts of level set Reeb graphs with equivalent topological structure. Then enhance these parts by adding their level set's Reeb graphs with respect to g as a secondary structure.*

3 Boundary-based computation of layered Reeb graphs

Several algorithms are described in the literature which compute the Reeb graph of a surface for a given surface description or the Reeb graph of a three-dimensional domain for a given volumetric description (like e.g. [8, 2, 6]). These algorithms typically allow for a rather general choice of defining functions. In this section we will restrict the defining functions such that the layered Reeb graph of a 3-manifold with boundary can be computed using only a boundary description of this manifold. This leads to computational advantages if the manifold is given in a boundary description, since no volumetric description has to be constructed. Additionally, it is thus possible to compute the layered Reeb graph of an unbounded manifold, for which the construction of a volumetric description may impose problems.

3.1 Feasible functions

The defining function for a Reeb graph has to meet two basic requirements:

(i) Firstly, it has to be possible to identify all the critical points, where structural changes in the level sets occur, using only function values on the boundary. This excludes the existence of local extrema or saddle points inside the considered manifold, see Figure 5.

(ii) Secondly, it should be possible to reconstruct the full structure of a level set knowing only the intersection of the level set with the manifold's boundary. This will become clearer in the following sections.

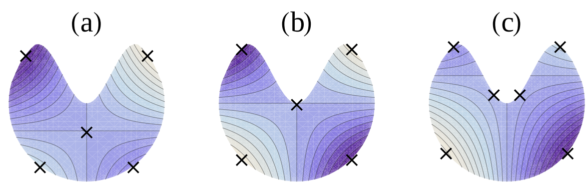

Figure 5: Contours of the function $(x, y) \to xy$ considered on planar 2-manifolds M with boundary. Critical points are marked by crosses. (a) forbidden: the origin is a critical point inside M. (b) forbidden: the origin is a critical point, but a regular point on the boundary. (c) allowed: all critical points on M are local minima or maxima on the boundary.

Feasibility of the first function

The first function f has to allow the computation of a Reeb graph of the 3-manifold M. For the Reeb graph of a 3-manifold, the considered level sets are, in general, surfaces.

Critical points (or curves) that are not induced by the boundary may occur where these level set surfaces touch each other or contract to a point. Since these cases are characterized by a zero gradient, we prescribe $\nabla f \neq 0$ to meet requirement (i).

The intersection of the level set surface with the manifold's boundary consists, in general, of closed curves. For requirement (ii), we need to be able to identify the relative positions of such boundary curves. Additionally, we need to know whether the inside or outside of such a curve is part of the considered level set. This can be deduced from the curve's orientation, provided that the boundary curves be computed in a certain orientation derived from the manifold's surface orientation. Using these two basic operations, the full structure of a level set can be reconstructed from the computed boundary curves. This is also a prerequisite of computing the Reeb graph of a level set component with respect to the second function.

This second requirement is fulfilled by functions for whose level sets a regular parametrization is known. Then, the two basic operations mentioned in the last paragraph can be reduced to operations in the level set's parameter domain. For instance, if another sufficiently smooth functions h is given such that $\nabla g \times \nabla h = \lambda \ \nabla f$ for some scalar field $\lambda > 0$, then g and h can be used as parameter functions on any level set of f, see Figure 6a.

Feasibility of the second function

The second function g is used to compute the Reeb graph of a level set surface of f. So, for every value c of f, the restriction g_c of g to the level set $f = c$ needs to meet the two basic requirements for the computation of a Reeb graph.

Consider requirement (i), i.e., that critical points are determined by the function values on the manifold's boundary. The level sets of g_c are now, in general, curves. They are formed by the intersection of level set surfaces of f and g, and thus have the tangential direction $\nabla f \times \nabla g$. Critical points inside the manifold occur if these level set curves touch or collapse to a point, which may happen only if the gradients of f and g are linearly dependent. So in order to meet the first requirement, we assume $\nabla f \times \nabla g \neq 0$.

For requirement (ii), we need to be able to reconstruct the structure of a level set from its boundary, e.g. in order to determine which level set component a critical point belongs to, while sweeping through the level sets of g_c. The level sets of g_c consist of segments on a curve, and their boundaries are simply points. We consider again the auxiliary function h, which is then monotonic along the level set of g_c, see Figure 6b. Then the curve endpoints can be sorted by their h-values, and intervals between them form level set components.

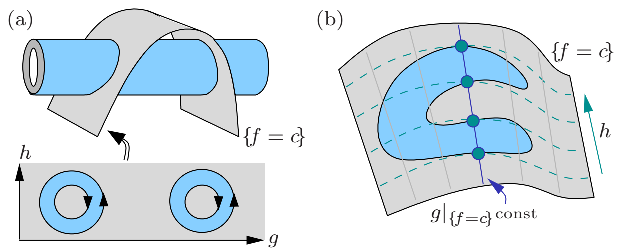

Figure 6: (a) Parametrization (g, h) of f-level sets. (b) Monotonic function h on level sets of $g|_{\{f=c\}}$.

Summary

Summing up we arrive at the following observation.

Lemma 1 *Assume $\nabla f \times \nabla g \neq 0$. Additionally, assume that another function h is available such that $\nabla h \times \nabla g = \lambda \ \nabla f$ for some scalar field $\lambda > 0$. Then, the layered Reeb graph with respect to f and g is determined by function values of f, g and h on the boundary of M.*

Proof. According to the observations in the previous sections, critical points of f and g_c on M that are not determined by function values on the boundary are excluded by $\nabla f \neq 0$ and $\nabla f \times \nabla g \neq 0$, respectively. Using the additional function h, the level sets of f can be equipped with a regular parametrization (g, h). Additionally, h is monotonic along the level sets of g_c. So all in all, requirements (i) and (ii) are met both for f and $g|_{\{f=c\}}$ □

3.2 Relation to Jacobi sets

Consider the critical points induced by the boundary. A boundary point p can only cause a change in the level set component if the functions' common level set touches the boundary in p without intersection. With N denoting the surface normal vector of M, these points are characterized by $(\nabla f \times \nabla g) \cdot N = 0$, since $\nabla f \times \nabla g$ is the tangent direction of the level set of f and g. This leads to a connection to the Jacobi set of the functions' restrictions to the manifold's surface, as defined in [3].

Definition 3 ([3]) *The* Jacobi set *of two smooth functions φ and ψ on a surface consists of all points with $\nabla\varphi \times \nabla\psi = 0$, with ∇ denoting the gradient operator on the surface.*

We get the following relation between Reeb spaces and Jacobi sets.

Lemma 2 *Let $\bar{f}$ and $\bar{g}$ denote the restrictions of f and g to the boundary of M. Then, the critical points of (f, g) which are induced by the boundary of M form the Jacobi set of $\bar{f}$ and $\bar{g}$.*

Proof. Let $\bar{\nabla}$ denote the gradient operator with respect to the boundary surface. For points on the Jacobi set, the gradients $\bar{\nabla}\bar{f}$ and $\bar{\nabla}\bar{g}$ are linearly dependent vectors in the boundary surface's tangent plane. Together with the surface normal vector N they define a plane ϵ, see Figure 7. Since ∇f and ∇g consist of $\bar{\nabla}\bar{f}$ and $\bar{\nabla}\bar{g}$, respectively, and components in the direction of N, they are also contained in ϵ. Therefore, $(\nabla f \times \nabla g) \cdot N = 0$, which characterizes a critical point of (f, g) induced by the boundary of M. The argumentation works analogously in the other direction. □

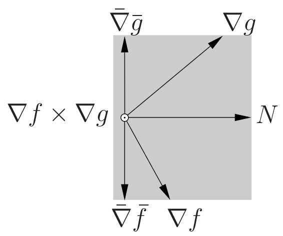

Figure 7: In critical points, gradients of f and g are coplanar with the surface normal vector N.

According to [3], the Jacobi sets of two functions defined on a surface are in general curves, so the critical points of f and g form curves on the manifold's surface. While sweeping through level sets of f, changes in the level set Reeb graphs thus occur at crossing points of these Jacobi curves, or at local extrema of the surface, which form monotonicity changes of Jacobi curves with respect to f. So the events to be considered in the sweep are crossing- and monotonicity-changing points of the Jacobi curves of $\overline{f}$ and $\overline{g}$.

3.3 Realization

The double-layered sweep algorithm sketched in the last paragraph of section 2.2 has been implemented for manifolds given by a triangular surface mesh, extending the algorithm for Reeb graphs as presented in [7]. We consider the piecewise linear approximations of the defining functions, as induced by the triangular mesh. If the mesh is fine enough compared to curvatures of the surface and the defining functions, the gradients of the piecewise linear approximation will approximate the smooth gradients sufficiently well.

In this setting, all critical points occur on edges of the surface mesh, so the Jacobi curves consist of mesh-edges. Using the criterion presented in [3], each edge is tested whether it is part of the Jacobi set or not. Additionally, boundary curves of the level sets of f are polygons, which can be computed and handled efficiently.

Conclusion

We presented a discrete representation for Reeb spaces of 3-manifolds with boundary with respect to two scalar-valued functions, the layered Reeb graph. Furthermore we introduced restrictions on the defining functions, which allow the layered Reeb graph to be computed from a boundary description of the manifold. In the next step, we seek to find a geometrically meaningful embedding of the Reeb space of a 3-manifold with respect to two functions into space. This leads to a topological skeleton of the 3-manifold, which promises to be more efficiently computable than, for example, the manifold's medial axis. The freedom in the choice of defining functions allows a customization of such a Reeb skeleton to give optimized results for given manifold.

References

[1] S. Biasotti, D. Giorgi, M. Spagnuolo, and B. Falcidieno. Reeb graphs for shape analysis and applications. *Theor. Computer Science*, 392(1-3):5–22, 2008.

[2] H. Doraiswamy and V. Natarajan. Efficient algorithms for computing Reeb graphs. *Computational Geometry*, 42(6-7):606–616, 2009.

[3] H. Edelsbrunner and J. Harer. Jacobi sets of multiple Morse functions. *Foundations of Computational Mathematics, Minneapolis 2002*, pages 35–57, 2004.

[4] H. Edelsbrunner, J. Harer, and A. K. Patel. Reeb spaces of piecewise linear mappings. In *Proc. Sympos. on Comput. Geom.*, pages 242–250. ACM, 2008.

[5] H. Edelsbrunner and M. Kerber. Alexander duality for functions: the persistent behavior of land and water and shore. In *Proc. Sympos. on Comput. Geom.*, pages 249–258. ACM, 2012.

[6] G. Patané, M. Spagnuolo, and B. Falcidieno. A minimal contouring approach to the computation of the Reeb graph. *IEEE Transactions on Visualization and Computer Graphics*, 15(4):583–595, 2009.

[7] B. Strodthoff, M. Schifko, and B. Jüttler. Horizontal decomposition of triangulated solids for the simulation of dip-coating processes. *Computer Aided Design*, 43:1891–1901, 2011.

[8] J. Tierny, A. Gyulassy, E. Simon, and V. Pascucci. Loop surgery for volumetric meshes: Reeb graphs reduced to contour trees. *IEEE Trans. on Visualization and Computer Graphics*, 15(6):1177–1184, 2009.