Projective and affine symmetries and equivalences of rational and polynomial surfaces

Michael Hauer, Bert Jüttler*, Josef Schicho

Institute of Applied Geometry / Research Institute for Symbolic Computation, Johannes Kepler University, Altenberger Str. 69, 4040 Linz, Austria

Abstract

It is known, that proper parameterizations of rational curves in reduced form are unique up to bilinear reparameterizations, i.e., projective transformations of its parameter domain. This observation has been used in a series of papers by Alcázar et al. to formulate algorithms for detecting Euclidean equivalences and symmetries as well as similarities. We generalize this approach to projective equivalences of rationally parametrized surfaces. More precisely, we observe that a birational base-point free parameterization of a surface is unique up to projective transformations of the domain. Furthermore, we use this insight to find all projective equivalences between two given surfaces. In particular, we formulate a polynomial system of equations whose solutions specify the projective equivalences, i.e., the reparameterizations associated with them.

Furthermore, we investigate how this system simplifies for the special case of affine equivalences for polynomial surfaces and how we can use our method to detect projective symmetries of surfaces. This method can be used for classifying the generic cases of quadratic surfaces.

Keywords: projective equivalences, symmetry detection, rational surface, polynomial system, linear reparameterization

1. Introduction

The detection of symmetries and equivalences of geometric objects is of interest in the fields of Computer Graphics, Computer Vision and Pattern Recognition and here several types of input data have been considered. On the one hand, for discrete input data (point sets, polygons and meshes) the detection is well understood and several efficient algorithms are available. Exact and also approximate congruences and symmetries of point sets, polygons and polytopes were studied in Computational Geometry, see e.g. Alt et al. [7], Huang and Cohen [14]. In recent years, research in Geometry Processing has focused on efficient algorithms for finding approximate congruences and symmetries of large point sets generated by 3D scans. The interested reader may consult the survey article by Mitra et al. [18] for further information.

On the other hand, the computation of symmetries and equivalences of algebraic curves, in particular of rational ones, experienced an increase of interest in the last years. These curves (and also surfaces) are important in geometric modeling, i.e., they are often used as a standard representation.

Using the implicit representation of planar curves, Lebmeir and Richter-Gebert [16] formulated a method for detecting congruences and symmetries. For curves of genus at least 2, Hess [13] described an algorithm for computing abstract isomorphisms (i.e., not necessarily projective ones)

*Corresponding author, phone +43732 2468 4080

Email addresses: `michael.hauer@jku.at` (Michael Hauer), `bert.juettler@jku.at` (Bert Jüttler), `josef.schicho@risc.jku.at` (Josef Schicho)

between algebraic curves based on their function fields. This algorithm is implemented in the computer algebra system Magma [8].

A substantial number of publications deals with the case of parametric curves. Most of these methods make use of Lemma 4.17 by Sendra et al. [21], which states that two proper parameterizations of the same rational curve are correlated by a linear rational reparameterization.

Recently Alcázar and his co-authors [1, 3, 4, 5, 6] published a series of papers on the detection of Euclidean equivalences and similarities of rationally parametrized planar and space curves. The first two publications make use of a coefficient-based method in the complex plane, whereas the latter ones employ invariants from differential geometry, such as the curvature and the torsion (possibly scaled for the detection of similarities). Sánchez-Reyes [20] used the Bernstein-Bézier basis to detect symmetries of polynomial curves using the control points.

Most of these publications concentrate on Euclidean and similarity transformations. In an earlier paper [12], we proposed an approach that deals with the general group of projective transformations for rationally parametrized curves and considers Euclidean, similarity and affine transformations as special cases. In the present article we generalize this approach to surfaces.

Symmetries in the sense of biregular automorphisms of algebraic surfaces are well understood theoretically, see [15, 23]. In the context of algebraic geometry, one considers nonsingular surfaces in projective spaces of arbitrary dimension; for surfaces in projective 3-space, which often have self-intersections and other singular curves, one may (theoretically) use the existence of a canonical resolution process that can be used to prove that the projective automorphism group of a singular surface in 3-space is a subgroup of the group of abstract automorphisms of its resolution. This approach, however, does not make it easy to compute projective symmetries or to decide projective equivalence. For this reason, we concentrate on parametric surfaces with known parametrization.

A first approach to symmetries of parametric surfaces has been presented by Alcázar and Hermoso [2], dealing with involutions of polynomially parametrized surfaces. There, the goal is to find involutions in the Euclidean group preserving the given surface.

The method described in our earlier paper [12], which is devoted to the curve case, is based on Lemma 4.17 by Sendra et al. [21]. In the present work we show that this result admits a generalization to proper birational and base-point-free parametrizations of the same surface (Theorem 4). Unless stated otherwise we consider the field of real numbers, i.e., all coefficients of the surfaces and all variables describing the transformation and reparameterization are given as real numbers. Note that our results are valid for other fields (such as complex numbers) as well, but in applications the real case is more interesting and we restrict ourselves to it. The generalization from surfaces to varieties of arbitrary dimension is possible as well, but for the sake of simplicity we restrict ourselves to surfaces. Besides, the practical interest in volumes and objects of even higher dimensions is less explicit.

The remainder of the paper is organized as follows. In Section 2 we fix our notation and recall some geometric and algebraic concepts. Section 3 is dedicated to the generalization of the Lemma about reparameterizations from curves to surfaces and we investigate how these reparameterizations act on the coefficients. In Section 4 we derive the method for detecting projective equivalences of rational surfaces. The detection of affine equivalences of polynomial surfaces is discussed as a special case. The following two sections provide examples and applications of our method, first to quadratically parametrized surfaces and then to higher degree ones. Finally we conclude the paper in Section 7 and discuss possibilities for future work.

2. Rational surfaces

We consider two-dimensional surfaces in the Euclidean d-space $\bar{E}^d$, which has been projectively closed (indicated by the bar) by adding points at infinity. Its points are represented by homogeneous coordinate vectors

$$\mathbf{x} = (x_0 : x_1 : \cdots : x_d)^T \in \mathbb{R}^{d+1} \backslash \{(0, \ldots, 0)\}.$$

Linearly dependent pairs of homogeneous coordinate vectors represent the same point, and this relation will be denoted by $\simeq$. More precisely, we write $\mathbf{x} \simeq \mathbf{y}$ if and only if there exists $\mu \neq 0$ such that $\mathbf{x} = \mu \mathbf{y}$.

Homogeneous coordinate vectors with $x_0 = 0$ represent points at infinity, and the collection of these points forms the hyperplane at infinity. All other points can be represented by Cartesian coordinates $\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_d)^T = (x_1/x_0, \ldots, x_d/x_0)^T$. Note that we use bold characters whenever we have a vector or a vector valued function.

We employ multi-indices (identified by bold font) consisting of three indices,

$$\mathbf{i} = (i_0, i_1, i_2).$$

We also use the notations

$$|\mathbf{i}| = i_0 + i_1 + i_2, \quad I^n = \{\mathbf{i} \in \mathbb{N}_0^3 \mid |\mathbf{i}| = n\}, \quad \mathbf{t}^{\mathbf{i}} = t_0^{i_0} t_1^{i_1} t_2^{i_2}$$

and the multinomial coefficients

$$\binom{|\mathbf{i}|}{\mathbf{i}} := \frac{|\mathbf{i}|!}{i_0! i_1! i_2!}.$$

In the remainder of the paper we investigate two surfaces $\mathcal{S}$ and $\mathcal{S}' \subset \bar{E}^d$ of degree n, which are considered as point sets. Both surfaces are given by birational base-point free parameterizations that map the projective plane to the surface, i.e.,

$$\mathbf{p} : P^2(\mathbb{R}) \to \mathcal{S} \subset \bar{E}^d, \quad \mathbf{t} \mapsto \mathbf{p}(\mathbf{t}) = (p_0(\mathbf{t}) : p_1(\mathbf{t}) : \cdots : p_d(\mathbf{t}))^T,$$

where

$$p_k(\mathbf{t}) = \sum_{\mathbf{i} \in I^n} c_{k,\mathbf{i}} \mathbf{t}^{\mathbf{i}} \quad \text{for} \quad k = 0, \ldots, d \tag{1}$$

are homogeneous polynomials of degree n in the parameter $\mathbf{t} = (t_0, t_1, t_2)$ that do not possess a common root over $\mathbb{C}$ and analogously for $\mathbf{p}'$. Note that the assumptions regarding birationality and base points are satisfied in the generic case, i.e., when considering surfaces with randomly generated coefficients. The surfaces are defined by $\binom{n+2}{2}$ coefficient vectors

$$\mathbf{c}_{\mathbf{i}} = (c_{0,\mathbf{i}}, c_{1,\mathbf{i}}, \ldots, c_{d,\mathbf{i}})^T, \quad \mathbf{i} \in I^n.$$

The parameterization and coefficients of the second surface $\mathcal{S}'$ are denoted by $\mathbf{p}'$ and $\mathbf{c}'_{\mathbf{i}}$, respectively. We will assume that neither of the two surfaces is contained in a hyperplane, in particular, that $\binom{n+2}{2} > d$. Consequently the coefficient matrix

$$C = (\mathbf{c}_{\mathbf{i}})_{\mathbf{i} \in I^n} \tag{2}$$

(and similarly the coefficient matrix C' of $\mathcal{S}'$) has rank $d + 1$.

We start with a lemma about the equivalence of the parameterizations.

Lemma 1. *Two rational base-point free parameterizations $\mathbf{p}(\mathbf{t})$ and $\mathbf{p}'(\mathbf{t})$ of a surface are equivalent, i.e. $\mathbf{p}(\mathbf{t}) \simeq \mathbf{p}'(\mathbf{t})$ holds for all $\mathbf{t} \in P^2(\mathbb{R})$, if and only if there exists a non-zero constant μ such that*

$$\mathbf{c}_{\mathbf{i}} = \mu \mathbf{c}'_{\mathbf{i}}, \quad \mathbf{i} \in I^n.$$

Proof. The equivalence of the parameterization of the two surfaces implies that there exists a rational function

$$\mu(\mathbf{t}) = \frac{\mu_1(\mathbf{t})}{\mu_0(\mathbf{t})} = \frac{p_0(\mathbf{t})}{p'_0(\mathbf{t})} = \frac{p_1(\mathbf{t})}{p'_1(\mathbf{t})} = \cdots = \frac{p_d(\mathbf{t})}{p'_d(\mathbf{t})}$$

where μ_0 are μ_1 are relatively prime polynomials, such that $\mathbf{p}(\mathbf{t}) = \mu(\mathbf{t})\mathbf{p}'(\mathbf{t})$. Consequently, the two rational surfaces satisfy

$$\mu_0(\mathbf{t})\mathbf{p}(\mathbf{t}) = \mu_1(\mathbf{t})\mathbf{p}'(\mathbf{t}).$$

This function is indeed a constant as

$$\mu_0 | \underbrace{\gcd(p'_0, p'_1, \ldots, p'_d)}_{=1} \quad \text{and} \quad \mu_1 | \underbrace{\gcd(p_0, p_1, \ldots, p_d)}_{=1},$$

since the parameterizations are base-point free.

The proof of the other implication is obvious. □

For any surface $\mathcal{S} \subset \bar{E}^d$, we define the graded coordinate ring $R := G(\mathcal{S})$ as the quotient ring of the polynomial ring $\mathbb{R}[x_0, \ldots, x_d]$ modulo the vanishing ideal of $\mathcal{S}$. As a real vectorspace, R is a direct sum of homogenous components R_i, $i \in \mathbb{N}$, where i is the degree, such that $R_i \cdot R_j \subset R_{i+j}$.

The Hilbert function $m \mapsto \dim(R_m)$ is "eventually polynomial", which means there is a polynomial $H \in \mathbb{Q}[m]$ whose value coincides with the value of the Hilbert function for all sufficiently large m. For instance, the Hilbert function of a surface of degree n in P^3 is equal to the polynomial $\binom{m+3}{3} - \binom{m+3-n}{3}$, for $m \geq n-3$.

For any integral domain R, the integral closure $\bar{R}$ is defined as the subset of its fraction field that satisfies an algebraic equation with leading coefficient 1 and all other coefficients in R. In particular R itself is contained in $\bar{R}$. If R is a graded ring, then its integral closure is generated by homogeneous fractions, i.e., by quotients of homogeneous elements. Therefore the integral closure of a graded ring is naturally graded, and the grading extends the grading of R. If the graded integral closure $\overline{G(\mathcal{S})}$ is isomorphic to the graded coordinate ring of another surface $\tilde{\mathcal{S}}$ of the same degree then $\tilde{\mathcal{S}}$ is also called a "projective normalization" of $\mathcal{S}$.

Example. Let $\mathcal{S} \subset \bar{E}^3$ be defined by the parameterization

$$(t_0 : t_1 : t_2) \mapsto (x_0 : x_1 : x_2 : x_3) = (t_0 t_2 : t_1^2 : t_1 t_2 : t_0^2).$$

Its vanishing ideal is generated by the polynomial $x_0^2 x_1 - x_2^2 x_3$. So, $G(\mathcal{S})$ is the quotient algebra of $\mathbb{R}[x_0, x_1, x_2, x_3]$ by the ideal generated by $x_0^2 x_1 - x_2^2 x_3$. The integral closure $\overline{G(\mathcal{S})}$ is generated by the classes of x_0, x_1, x_2, x_3 and by the fraction $x_4 := \frac{x_2 x_3}{x_0} = \frac{x_0 x_1}{x_2}$ which fulfils the integral equation $x_4^2 - x_1 x_3 = 0$. It is isomorphic to the quotient ring of $\mathbb{R}[x_0, x_1, x_2, x_3, x_4]$ by the ideal generated by the polynomials $x_4^2 - x_1 x_3, x_0 x_4 - x_2 x_3, x_0 x_1 - x_2 x_4$ and which also contains the original equation $x_0^2 x_1 - x_2^2 x_3$. The surface $\tilde{\mathcal{S}} \subset \bar{E}^4$ defined by the equations $x_4^2 - x_1 x_3 = x_0 x_4 - x_2 x_3 = x_0 x_1 - x_2 x_4 = 0$ is a projective normalization.

In general, the ring $\overline{G(\mathcal{S})}$ needs not be isomorphic to the coordinate ring of a surface in projective space: it is possible that not all its generators have degree 1. But we will show in Section 3 that surfaces with a base-point free and birational parameterization do have a projective normalization in projective space.

Any parameterization of $\mathbf{p} : P^2 \to \mathcal{S}$ of degree n induces a ring homomorphism $p^* : G(\mathcal{S}) \to G(P^2)$, where $G(P^2) = \mathbb{R}[t_0, t_1, t_2]$ is the graded coordinate ring of P^2. It maps $x_0, \ldots, x_d$ to $p_0, \ldots, p_d$. The map maps homogeneous elements to homogeneous elements, but the degree gets multiplied by n.

The ring homomorphism p^* can be extended to its integral closure. Because $G(P^2)$ is already integrally closed, this gives a ring homomorphism $\overline{G(\mathcal{S})} \to G(P^2)$, denoted also by p^*.

Example continued. The ring homomorphism p^* for the surface in the previous example maps x_0 to $t_0 t_2$, x_1 to t_1^2, x_2 to $t_1 t_2$, and x_3 to t_0^2. The image of the new element x_4 is $\frac{t_0 t_2 t_1^2}{t_1 t_2} = t_0 t_1$.

3. Reparameterizations

In this section we will show that two proper birational and base-point-free parameterizations of the same rational surface are correlated by a linear rational reparameterization (Theorem 4). The main point is to show the reparametrization obtained by composing one parametrization with the inverse of the second, which is a priori only birational, is even biregular. Recall that a birational map is called biregular if and only if both the map and its birational inverse are everywhere defined.

The Veronese surface $V_n \subset P^{\frac{n(n+3)}{2}}$ is defined by a parameterization of degree n consisting of all powers of $\mathbf{t}$ of degree n (there are $\binom{n+2}{2} = \frac{n(n+3)}{2} + 1$ of them), i.e.,

$$p_{k(\mathbf{i})} = \mathbf{t}^{\mathbf{i}}, \quad k(\mathbf{i}) \text{ is a numbering of } I^n.$$

Its graded coordinate ring is isomorphic to the subring $G_n(P^2)$ of $G(P^2)$ generated by all homogeneous elements of degree divisible by n. We will show that the Veronese surface is a projective normalization of any surface that has a birational base-point free parameterization: the statement is equivalent to Lemma 2.

We need another concept from elimination theory: for three general homogeneous polynomials $f_0, f_1, f_2 \in \mathbb{R}[t_0, t_1, t_2]$ of the same degree m, the "Macaulay resultant" M_{f_0,f_1,f_2} is defined as a polynomial in the $3\binom{m+2}{2}$ coefficients of f_0, f_1, f_2. Its total degree is $3m^2$, but there are various other homogenity properties. Here we need that it is weighted homogeneous where the weight of each coefficient is the exponent of t_0 in the corresponding power products. The main property of the Macaulay resultant is that it vanishes if and only if f_0, f_1, f_2 have a nontrivial common zero.

Lemma 2. *If $\mathbf{p}$ is a birational base-point free parameterization, then the image of $\overline{G(\mathcal{S})}$ under $p^* : \overline{G(\mathcal{S})} \to G(P^2)$ is the subring $G_n(P^2)$.*

Proof. Let $R := p^*(\overline{G(\mathcal{S})})$. Because p^* multiplies the degree by n, it follows that $R \subset G_n(P^2)$.

For the converse, we first show that $G_n(P^2)$ is contained in the fraction field of R. Birationality means that there exists a rational inverse $(x_0 : \cdots : x_d) \to (g_0(x_0, \ldots, x_d) : g_1(x_0, \ldots, x_d) : g_2(x_0, \ldots, x_d))$, with $g_0, g_1, g_2 \in G(\mathcal{S})$, which we can assume to be homogeneous, of degree, say, M. Because composition with the inverse is the identity, there exists a polynomial $h \in \mathbb{R}[t_0, t_1, t_2]$ such that $p^*(g_i) = t_i h$ for $i = 0, 1, 2$. (The degree of h must be $Mn - 1$.) For any polynomial $f \in \mathbb{R}[t_0, t_1, t_2]$ of degree n, the fraction f/p_0 is in the fraction field of R, because

$$\frac{f}{p_0} = \frac{f(ht_0, ht_1, ht_2)}{p_0(ht_0, ht_1, ht_2)} = \frac{p^*(f(g_0, g_1, g_2))}{p^*(p_0(g_0, g_1, g_2))}.$$

Therefore f is in the fraction field of R.

In order to show that any element in $G_n(P^2)$ satisfies an integral equation over R, we show (slightly stronger) that t_0, t_1, and t_2 satisfy an integral equation. This is sufficient, because it is well-known and also easy to prove that if a and b satisfy integral equations, then so do their sum $a + b$ and their product ab. First, we find three polynomials q_0, q_1, q_2 in the linear span L of $p_0, \ldots, p_d$ without any nontrivial common zero. We choose $q_0 := p_0$. Then we choose q_1 among $p_1, \ldots, p_d$ relatively prime to p_0 – this is possible because otherwise all p_i, $i = 0, \ldots, d$, would have a common factor. Note that q_0 and q_1 have finitely many common zeroes (namely n^2 when counted with multiplicities). For any intersection point x, the subspace of all polynomials in L vanishing at x is proper (of codimension 1). So we choose q_2 outside the union of all these finitely many subspaces.

Let y_0, y_1, y_2 be three new variables. Let M_0 be the Macaulay-resultant of the three polynomials $q_0(s_0t_0, s_1, s_2) - y_0s_0^n, q_1(s_0t_0, s_1, s_2) - y_1s_0^n, q_2(s_0t_0, s_1, s_2) - y_2s_0^n$, considered as homogeneous polynomials in s_0, s_1, s_2 of degree n. Then M_0 is a polynomial in y_0, y_1, y_2, t_0, weighted homogeneous of degree n^2 where the weight of t_0 is 1 and the weights of y_0, y_1, y_2 are n. Because q_0, q_1, q_2 do not have any common zeroes, $M_0(0, 0, 0, 1) \neq 0$. After division by this coefficient, we get an integral equation for t_0 of degree n^2 with coefficients in $\mathbb{R}[y_0, y_1, y_2]$. Now $M_0(p_0, p_1, p_2, t_n) = 0$ because apparently $q_0(s_0t_0, s_1, s_2) - p_0s_0^n, q_1(s_0t_0, s_1, s_2) - p_1s_0^n, q_2(s_0t_0, s_1, s_2) - p_2s_0^n$ do have a common zero $(s_0 : s_1 : s_2) := (t_0 : t_1 : t_2)$. Hence after substituting y_0, y_1, y_2 by p_0, p_1, p_2, we get an integral equation for t_0 with coefficients in R. Similarly, t_1 and t_2 satisfy such integral equations. It follows that $G_n(P^2) \subset R$. □

The Veronese surface V_n is isomorphic to P^2 as an algebraic variety: the birational parameterization $\mathbf{v}_n : P^2 \to V_n$ given by the power functions

$$\mathbf{t} \mapsto (x_{\mathbf{i}})_{\mathbf{i} \in I^n} = (\mathbf{t}^{\mathbf{i}})_{\mathbf{i} \in I^n}$$

has the inverse

$$(x_{\mathbf{i}})_{\mathbf{i}\in I^n} \mapsto \mathbf{t} = (x_{(a+1,b,c)} : x_{(a,b+1,c)} : x_{(a,b,c+1)})$$

for any non-negative integers a, b, c such that $a+b+c = n-1$. Both $\mathbf{v}_n$ and $(\mathbf{v}_n)^{-1}$ are everywhere defined, hence they provide an isomorphism in the category of algebraic varieties.

We need to investigate the degree-preserving ring automorphisms of $G_n(P^2)$ fixing $\mathbb{R}$ element-wise (also called graded $\mathbb{R}$-automorphisms). Examples of such automorphisms are "substitution automorphisms" obtained by substituting for t_0, t_1, t_2 three linear independent linear forms.

Lemma 3. *Any graded $\mathbb{R}$-automorphism of $G_n(P^2)$ is a substitution automorphism.*

Proof. Any graded $\mathbb{R}$-automorphism induces a birational map $\mathbf{a} : V_n \to V_n$ that is everywhere defined (even more, an automorphism of V_n as an algebraic variety). Then the birational map $\mathbf{b} := (\mathbf{v}_n)^{-1} \circ \mathbf{a} \circ \mathbf{v}_n : P^2 \to P^2$ is also everywhere defined. Then $\mathbf{b}$, as a rational map from P^2 to itself, can be defined by a triple of polynomials $(b_0, b_1, b_2) \in G(P^2)$, homogeneous of the same degree m. Since the map is everywhere defined, the three polynomials do not have a nontrivial common zero. Then the preimage of a generic point is the intersection of two generic linear combinations of b_0, b_1, b_2. This is a union of m^2 points. But the map is birational, hence $m^2 = 1$ and the map $\mathbf{b}$ is linear. □

Theorem 4. *Let $\mathbf{p}, \mathbf{p}' : P^2 \to \mathcal{S}$ be two parameterizations of degree n of the same surface $\mathcal{S} \subset P^d$. Assume that both parameterizations are base-point free and birational. Then there exists a linear reparameterization $\mathbf{r} : P^2 \to P^2$ such that $\mathbf{p}' = \mathbf{p} \circ \mathbf{r}$.*

Proof. By Lemma 2, the parameterizations induce two ring isomorphisms $p^*, p'^* : \overline{G(\mathcal{S})} \to G_n(P^2)$. The ring automorphism $p'^* \circ (p^*)^{-1} : G_n(P^2) \to G_n(P^2)$ preserves the degree, hence it is a substitution automorphism by Lemma 3. The linear substitution $\mathbf{r} : P^2 \to P^2$ inducing $p'^* \circ (p^*)^{-1}$ then satisfies the desired equality $\mathbf{p}' = \mathbf{p} \circ \mathbf{r}$. □

We represent this linear reparameterization $\mathbf{r}$ by a transformation matrix α

$$\mathbf{r}(\mathbf{t}) = \underbrace{\begin{pmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} \end{pmatrix}}_{=\alpha} \mathbf{t} = \begin{pmatrix} \alpha_{00}t_0 + \alpha_{01}t_1 + \alpha_{02}t_2 \\ \alpha_{10}t_0 + \alpha_{11}t_1 + \alpha_{12}t_2 \\ \alpha_{20}t_0 + \alpha_{21}t_1 + \alpha_{22}t_2 \end{pmatrix}.$$

Finally, we investigate the influence of such a reparameterization on the coefficients of a degree n surface. The following lemma states that the coefficients of the reparametrized surface are given as a linear combination of the coefficients of the original one, where the reparameterization determines the influence of the linear factors.

Lemma 5. *The reparametrized surface*

$$\hat{\mathbf{p}}(\mathbf{t}) = (\mathbf{p} \circ \mathbf{r})(\mathbf{t}) = \sum_{\mathbf{j}\in I^n} \hat{\mathbf{c}}_{\mathbf{j}}(\alpha)\mathbf{t}^{\mathbf{j}}$$

has the coefficients

$$\hat{\mathbf{c}}_{\mathbf{j}}(\alpha) = \sum_{\mathbf{i}\in I^n} \mathbf{c}_{\mathbf{i}} \sum_{\substack{(|\mathbf{k}|,|\boldsymbol{\ell}|,|\mathbf{m}|)=\mathbf{i} \\ \mathbf{k}+\boldsymbol{\ell}+\mathbf{m}=\mathbf{j}}} \binom{|\mathbf{k}|}{\mathbf{k}} \binom{|\boldsymbol{\ell}|}{\boldsymbol{\ell}} \binom{|\mathbf{m}|}{\mathbf{m}} \alpha_{00}^{k_0}\alpha_{10}^{\ell_0}\alpha_{20}^{m_0}\alpha_{01}^{k_1}\alpha_{11}^{\ell_1}\alpha_{21}^{m_1}\alpha_{02}^{k_2}\alpha_{12}^{\ell_2}\alpha_{22}^{m_2}. \tag{3}$$

Proof. A computation gives that

$$(\mathbf{p} \circ \mathbf{r})(\mathbf{t}) = \sum_{\mathbf{i}\in I^n} \mathbf{c}_{\mathbf{i}} \quad (\alpha_{00}t_0 + \alpha_{01}t_1 + \alpha_{02}t_2)^{i_0}(\alpha_{10}t_0 + \alpha_{11}t_1 + \alpha_{12}t_2)^{i_1}$$
$$(\alpha_{20}t_0 + \alpha_{21}t_1 + \alpha_{22}t_2)^{i_2}$$

$$= \sum_{\mathbf{i}\in I^n} \mathbf{c}_\mathbf{i} \left(\sum_{|\mathbf{k}|=i_0} \binom{i_0}{\mathbf{k}} \alpha_{00}^{k_0} t_0^{k_0} \alpha_{01}^{k_1} t_1^{k_1} \alpha_{02}^{k_2} t_2^{k_2} \right) \left(\sum_{|\boldsymbol{\ell}|=i_1} \binom{i_1}{\boldsymbol{\ell}} \alpha_{10}^{\ell_0} t_0^{\ell_0} \alpha_{11}^{\ell_1} t_1^{\ell_1} \alpha_{12}^{\ell_2} t_2^{\ell_2} \right)$$

$$\left(\sum_{|\mathbf{m}|=i_2} \binom{i_2}{\mathbf{m}} \alpha_{20}^{m_0} t_0^{m_0} \alpha_{21}^{m_1} t_1^{m_1} \alpha_{22}^{m_2} t_2^{m_2} \right)$$

$$= \sum_{\mathbf{i}\in I^n} \mathbf{c}_\mathbf{i} \sum_{(|\mathbf{k}|,|\boldsymbol{\ell}|,|\mathbf{m}|)=\mathbf{i}} \binom{i_0}{\mathbf{k}} \binom{i_1}{\boldsymbol{\ell}} \binom{i_2}{\mathbf{m}} \alpha_{00}^{k_0} \alpha_{10}^{\ell_0} \alpha_{20}^{m_0} \alpha_{01}^{k_1} \alpha_{11}^{\ell_1} \alpha_{21}^{m_1} \alpha_{02}^{k_2} \alpha_{12}^{\ell_2} \alpha_{22}^{m_2}$$

$$t_0^{k_0+\ell_0+m_0} t_1^{k_1+\ell_1+m_1} t_2^{k_2+\ell_2+m_2}$$

$$= \sum_{\mathbf{i}\in I^n} \mathbf{c}_\mathbf{i} \sum_{\mathbf{j}\in I^n} \mathbf{t}^\mathbf{j} \sum_{\substack{(|\mathbf{k}|,|\boldsymbol{\ell}|,|\mathbf{m}|)=\mathbf{i} \\ \mathbf{k}+\boldsymbol{\ell}+\mathbf{m}=\mathbf{j}}} \binom{i_0}{\mathbf{k}} \binom{i_1}{\boldsymbol{\ell}} \binom{i_2}{\mathbf{m}} \alpha_{00}^{k_0} \alpha_{10}^{\ell_0} \alpha_{20}^{m_0} \alpha_{01}^{k_1} \alpha_{11}^{\ell_1} \alpha_{21}^{m_1} \alpha_{02}^{k_2} \alpha_{12}^{\ell_2} \alpha_{22}^{m_2}$$

$$= \sum_{\mathbf{j}\in I^n} \mathbf{t}^\mathbf{j} \sum_{\mathbf{i}\in I^n} \mathbf{c}_\mathbf{i} \sum_{\substack{(|\mathbf{k}|,|\boldsymbol{\ell}|,|\mathbf{m}|)=\mathbf{i} \\ \mathbf{k}+\boldsymbol{\ell}+\mathbf{m}=\mathbf{j}}} \binom{|\mathbf{k}|}{\mathbf{k}} \binom{|\boldsymbol{\ell}|}{\boldsymbol{\ell}} \binom{|\mathbf{m}|}{\mathbf{m}} \alpha_{00}^{k_0} \alpha_{10}^{\ell_0} \alpha_{20}^{m_0} \alpha_{01}^{k_1} \alpha_{11}^{\ell_1} \alpha_{21}^{m_1} \alpha_{02}^{k_2} \alpha_{12}^{\ell_2} \alpha_{22}^{m_2}$$

Comparing the coefficients confirms (3). □

4. Equivalences

Recall that the use of homogeneous coordinates allows to represent any regular projective transformation f by a matrix multiplication

$$f : \bar{E}^d \to \bar{E}^d : \mathbf{x} \mapsto f(\mathbf{x}) = M\mathbf{x},$$

where $M = (m_{ij})_{i,j=0,\dots,d}$ is a non-singular real matrix. If

$$m_{00} \neq 0 \quad \text{and} \quad m_{01} = \cdots = m_{0d} = 0, \tag{4}$$

then f is an affine transformation. If additionally the matrix

$$A = \left(\frac{m_{ij}}{m_{00}} \right)_{i,j=1,\dots,d},$$

is orthogonal $A^T A = I$, we have an Euclidean transformation, i.e., the composition of a rotation, a translation and possibly a reflection. If $A^T A = \lambda I$ with $\lambda \in \mathbb{R}^+$, some additional scaling may be involved, and A describes a similarity transformation. All these transformations are special cases of projective transformations. We consider pairs of surfaces, that are related by projective transformations.

Two surfaces $\mathcal{S}$ and $\mathcal{S}' \subset \bar{E}^d$ are said to be *projectively/affinely equivalent* if there exists a regular projective/affine transformation f such that $\mathcal{S}' = f(\mathcal{S})$. Furthermore, $\mathcal{S}$ is said to possess a *projective/affine symmetry* if there exists a regular projective/affine transformation f, different from the identity, such that $\mathcal{S} = f(\mathcal{S})$.

If $\mathcal{S}'$ is projectively equivalent to $\mathcal{S}$, then $\mathcal{S}$ is also projectively equivalent to $\mathcal{S}'$, as the projective transformation f is assumed to be regular. Moreover, each surface is projectively equivalent to itself by the identity map. The transitivity of the relation is implied by the group structure of regular projective mappings. Therefore, the projective equivalence defines an equivalence relation.

We identify projective equivalences of surfaces by analyzing whether the matrices defined by the coefficients are related by a projective transformation. Furthermore we investigate the special case of affine equivalences of polynomial surfaces.

4.1. Projective equivalences

Proposition 6. *Let $\mathcal{S}$ and $\mathcal{S}'$ be rational surfaces of total degree n such that $\binom{n+2}{2} > d$. Let them be given by birational base-point free parameterizations $\mathbf{p}(\mathbf{t})$ and $\mathbf{p}'(\mathbf{t})$, which are defined by their coefficient matrices C and C' of rank $d+1$, see Equation (2). The two surfaces are projectively equivalent if and only if there exists a regular projective transformation matrix M and a projective transformation α of the real plane, such that C and C' satisfy*

$$MC' = \hat{C}(\alpha). \tag{5}$$

Proof. On the one hand, the conditions (5) imply that the two surfaces are projectively equivalent. On the other hand, we consider two projectively equivalent surfaces $\mathcal{S}'$ and $\mathcal{S}$. There exists a projective transformation f with the matrix M such that

$$f(\mathcal{S}') = \mathcal{S}.$$

We define $\mathbf{q}(\mathbf{t}) = M\mathbf{p}'(\mathbf{t})$. Consequently $\mathbf{q}(\mathbf{t})$ and $\mathbf{p}(\mathbf{t})$ are two birational base-point free parameterizations of the same surface $\mathcal{S}$. According to Theorem 4 there is a linear rational reparameterization $\mathbf{r}(\mathbf{t})$ – and hence an associated projective transformation α – such that

$$\mathbf{q}(\mathbf{t}) \simeq (\mathbf{p} \circ \mathbf{r})(\mathbf{t}). \tag{6}$$

Thus using the Equations (1) and (6) and Lemma 5 we obtain that

$$\sum_{\mathbf{i} \in I^n} M\mathbf{c}'_{\mathbf{i}}\mathbf{t}^{\mathbf{i}} = M\mathbf{p}'(\mathbf{t}) = \mathbf{q}(\mathbf{t}) \simeq (\mathbf{p} \circ \mathbf{r})(\mathbf{t}) = \sum_{\mathbf{i} \in I^n} \hat{\mathbf{c}}_{\mathbf{i}}(\alpha)\mathbf{t}^{\mathbf{i}}.$$

comparing the coefficients and using Lemma 1 gives

$$MC' \simeq \hat{C}(\alpha).$$

Finally we put the constant μ of the homogeneous coordinates into M which confirms Equation (5). □

Hence in order to detect projective equivalences of two surfaces we have to solve Equation (5). Unfortunately it is impracticable to solve this system directly, as even for small degree ($n = 2$) surfaces in space ($d = 3$) the computation of the corresponding Gröbner basis takes quite long and the size of the coefficients grows very fast. Also other standard methods do not lead to acceptable computation times of solving this polynomial system in $(d + 1)^2 + 9$ unknowns in M and α.

We observe that the system (5) has a special structure, i.e., it is linear in the unknowns of M and the right hand side consists of homogeneous polynomials of degree n in α. We rewrite this equation in order to eliminate the unknowns in M and to obtain a system of polynomial equations of degree n in the unknowns α.

Proposition 7. *Let $\mathcal{S}$ and $\mathcal{S}'$ be as in Proposition 6. Let*

$$\mathbf{b}^\ell = (b^\ell_{\mathbf{i}})_{\mathbf{i} \in I^n}, \quad \ell = 1, \ldots, \binom{n+2}{2} - d - 1$$

be basis vectors spanning the kernel of C'.

The two surfaces are projectively equivalent if and only if there is a linear reparameterization determined by a regular matrix α such that

$$\sum_{\mathbf{i} \in I^n} \hat{c}_{k,\mathbf{i}}(\alpha) b^\ell_{\mathbf{i}} = 0, \quad k = 0, \ldots, d \quad \ell = 1, \ldots, \binom{n+2}{2} - d - 1 \tag{7}$$

is satisfied, where the coefficients $\hat{c}_{k,\mathbf{i}}(\alpha)$ are given in (3).

Proof. The coefficient matrix C' has rank $d+1$, hence its kernel has dimension $\binom{n+2}{2} - d - 1$. This confirms the existence of the kernel basis vectors. We show that Equation (7) is equivalent to condition (5).

First, Equation (5) ensures that the kernel of the matrix C' is contained in the kernel of the matrix $\hat{C}(\alpha)$ and this proves (7).

For the other direction we have that Equation (7) implies that the kernel of $\hat{C}(\alpha)$ contains the kernel of C'. This implies that the space spanned by the row vectors of $\hat{C}(\alpha)$ is contained in the space spanned by the row vectors of C', since these spaces are the orthogonal complement of the kernels. This proves the existence of the matrix M in (5). Its regularity is implied by the regularity of α and the assumption on the coefficient matrices which ensure that both $\hat{C}(\alpha)$ and C' have rank $d+1$. $\square$

To ensure the regularity of α we add the unknown u and the equation

$$\det(\alpha)u = 1 \tag{8}$$

to our system. As the reparameterization of the parameter domain, which is the projective plane, is only given up to a non-zero multiplicative constant we normalize α by setting the first nonzero coefficient in the first row to 1, which leads to 3 cases.

We solve the system (7), (8) for the different normalizations. These systems consist of $(d+1)\left(\binom{n+2}{2} - d - 1\right) + 1$ equations in at most 9 unknowns for α and u. Solving these systems is the most time-consuming part of our method.

For any reparameterization α the corresponding projective transformation M is obtained simply by solving the linear system of equations

$$M\mathbf{c}'_{\mathbf{i}^{(\ell)}} = \hat{\mathbf{c}}_{\mathbf{i}^{(\ell)}}(\alpha), \quad \ell = 0, \ldots, d,$$

for the $(d+1)^2$ unknown elements of M, where we can choose any $d+1$ linear independent coefficient vectors $\mathbf{c}'_{\mathbf{i}^{(\ell)}}$. Here the computational effort is negligible.

The specific type of the equivalence can be found by investigating the properties of the transformation matrix M. More precisely, it is an affine equivalence if the elements satisfy

$$m_{0i} = 0, \quad \text{for} \quad i = 1, \ldots, d.$$

It is a similarity (or even a congruence transformation) if additionally the condition

$$A^T A = \lambda I \quad \text{with} \quad A = \left(\frac{m_{ij}}{m_{00}}\right)_{i,j=1,\ldots,d}$$

is fulfilled, where I is the $d \times d$ identity matrix (and the factor even satisfies $\lambda = 1$ for congruence transformations).

When applied to pairs $(\mathcal{S}, \mathcal{S})$ of identical surfaces, the method allows us to identify all projective symmetries. This includes all affine or Euclidean symmetries, which are found by analyzing the properties of the corresponding transformation matrix M, analogously to the discussion above. In the case of symmetry detection the identity is always a solution of the system.

Remark 8. In the numerical examples we tried several computer algebra systems for computing the Gröbner basis and solving the system, i.e. we implemented our method in Singular 4-0-2 [10], Mathematica® Version 11 [22] and Maple™ 2017 [17]. We obtained the best results by computing the Gröbner basis in Maple, which uses the C library MGb, and solving the emerging system with Mathematica. The MGb library takes advantage of the fact that our system is sparse and well structured, i.e., it is homogeneous of degree n in the non constant parts.

4.2. Affine equivalences

If we are not looking for projective equivalences, but for affine ones, we add additional equations, which reduce the computation time.

Table 1: Specifications of the polynomial systems

	# systems	# equations	# unknowns
proj. equivalence of rational surfaces	3	$\left[\binom{n+2}{2} - d - 1\right](d+1) + 1$	9/8/7
aff. equivalence of rational surfaces	3	$\left[\binom{n+2}{2} - d - 1\right](d+1) + \binom{n+2}{2} + 1$	10/9/8
aff. equivalence of polynomial surfaces	1	$\left[\binom{n+2}{2} - d - 1\right](d+1) + 1$	7

Corollary 9. *Let $\mathcal{S}$ and $\mathcal{S}'$ be as in Proposition 6. If $\mathcal{S}$ and $\mathcal{S}'$ are two affinely equivalent rational surfaces, there is a linear reparameterization, given by the matrix α such that (7), (8) and*

$$\omega c'_{0,\mathbf{i}} = \hat{c}_{0,\mathbf{i}}(\alpha), \quad \mathbf{i} \in I^n, \tag{9}$$

with $\omega \neq 0 \in \mathbb{R}$, are satisfied.

Proof. We note that as M is a regular affine transformation we have $m_{00} \neq 0$ and $m_{0k} = 0$ for $k = 1, \ldots, d$. Hence the first row of Equation (5) gives (9) with $\omega = m_{00}$. □

In this case, the number of unknowns is reduced when we consider only polynomial surfaces. This is due to the fact that two of the elements of the reparameterization matrix α are equal to zero.

Proposition 10. *Let $\mathcal{S}$ and $\mathcal{S}'$ be as in Proposition 6. If $\mathcal{S}$ and $\mathcal{S}'$ are two affinely equivalent polynomial surfaces the reparameterization is an affine linear parameter transformation, i.e. $\alpha_{01} = \alpha_{02} = 0$.*

Proof. As both surfaces are polynomial $c_{0,(n,0,0)} \neq 0 \neq c'_{0,(n,0,0)}$ and all other coefficients of the first row in the matrices C and C' are equal to 0. Hence by Equation (3) and Corollary 9 we have for $\mathbf{i} = (0, n, 0)$

$$0 = m_{00}c'_{0,(0,n,0)} = \hat{c}_{0,(0,n,0)} = c_{0,(n,0,0)}\alpha_{01}^n$$

and analogously for $\mathbf{i} = (0, 0, n)$

$$0 = m_{00}c'_{0,(0,0,n)} = \hat{c}_{0,(0,0,n)} = c_{0,(n,0,0)}\alpha_{02}^n,$$

which confirms the statement. □

We note that the conditions in Equation (9) are automatically fulfilled for affinely invariant polynomial surfaces. In particular for $\mathbf{i} = (n, 0, 0)$ we have

$$\omega c'_{0,\mathbf{i}} = \alpha_{00}^n c_{0,\mathbf{i}}$$

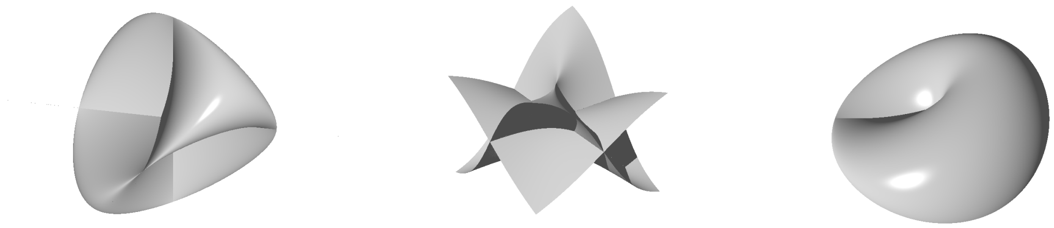
which is satisfied by a suitable choice of ω and in all other equations both sides of the equation evaluate to zero.

Consequently, for the special case of affine equivalences of polynomial surfaces we only have to consider one normalization, i.e. $\alpha_{00} = 1$ and $\alpha_{01} = \alpha_{02} = 0$. Hence the number of unknowns decreases from 9 to 7, since we need to compute only u, $\alpha_{1,j}$ and $\alpha_{2,j}$, for $j = 0, 1, 2$. The normalization also leads to simplifications in Equation (3). Similar to the projective rational case, the system we solve consists the equations (7) and (8).

We summarize the specifications of the polynomial systems in Table 1.

5. Application to quadratically parametrizable surfaces

The computation time presented in the following two sections refer to our implementation using Maple to compute the Gröbner basis and Mathematica for solving the emerging system, see Remark 8. All computations were performed on an Intel Core i7 PC, with 3.4 GHz and 32 GB RAM.



Σ_1 Σ_2 Σ_3

Figure 1: The general cases of quadratic rational surfaces

Table 2: Projective classification of 100 randomly generated surfaces with coefficients $|c_{(k,\mathbf{i})}| \leq 100$

Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	not equivalent to $\Sigma_1, \ldots, \Sigma_6$
27	12	61	0	0	0	0

5.1. Projective equivalences of rational surfaces

Nondegenerated quadratically parameterizable surfaces in space can be classified over the real numbers into 12 projective equivalence classes. Three of them belong to the well known and thouroughly studied quadrics, i.e., there exist the three projective classes of oval quadrics, ring quadrics and cones. The remaining nine classes were discussed in Coffman et al. [9] who also give a normal form for each class. Degen [11] proposed a different approach for classifying triangular surfaces by identifying them as projections of the Veronese surface from $P^5(\mathbb{R})$ into $P^3(\mathbb{R})$. We follow the notation by Coffman et al. [9] and denote these surfaces by Σ_1 to Σ_9. The implicit equations of the surfaces Σ_1 to Σ_6 are quartic, and the parameterizations are base-point free. In contrast to this, the quadrics and the surfaces Σ_7, Σ_8 and Σ_9 have implicit equations of lower degrees and their parameterizations are not base-point free.

In our first experiment we want to confirm experimentally that the generic cases are the first three. In order to do that, we randomly generate 100 parameterizations of quadratic surfaces by randomly choosing integer coefficients with an absolute value less then 100, i.e., $|c_{(k,\mathbf{i})}| \leq 100$. We then check whether they belong to one of the classes Σ_1 to Σ_6. If they are not equivalent to one of these classes they either possess a base-point or they are contained in a hyperplane.

We present our results in Table 2, which show that Σ_1, Σ_2 and Σ_3 are the generic cases as none of the other classes occured. Hence, for the general case our algorithm provides a simple alternative method to classify a given surface. Moreover, we obtain the reparameterizations and projective transformations that transform the input surface into the normal form.

The second question we want to address is how many projective symmetries the surfaces, that fulfil our assumptions, possess and whether further symmetries within one class (or equivalences between the classes) exist if we also allow complex solutions. We applied our method for symmetry detection on the normal forms of Σ_1 to Σ_6 and every solution describes a projective symmetry. In the upper part of Table 3 we list the number of symmetries. The general cases Σ_1 to Σ_3 possess a discrete number of symmetries, whereas the solutions for Σ_4 to Σ_6 depend on one parameter and hence there are infinitely many of them.

Coffman et al. [9] already mentioned that Σ_1, Σ_2 and Σ_3 belong to the same class if one considers complex projective transformations, and so do Σ_4 and Σ_5. We could verify this as well, see the lower part in Table 3. We did not find any further relations of different classes.

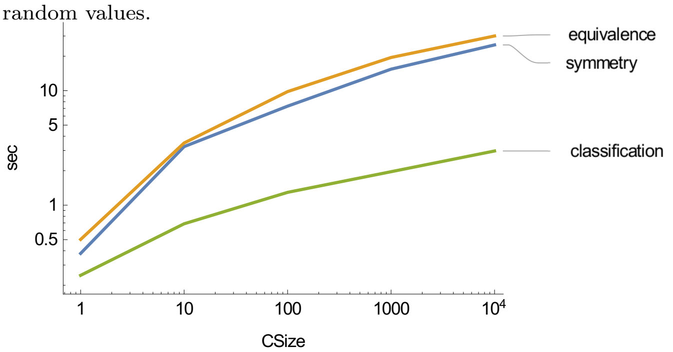
Due to the known classification we have two different ways of detecting equivalences of two surfaces. Firstly, one may apply our method directly to the two surfaces. Secondly, one may classify both surfaces and check whether they belong to the same class.

It is interesting to analyze which of the two approaches is faster, and how this depends on the size of the coefficients. To investigate this question we generated again surfaces with random coefficients smaller than some given constant, $|c_{(k,\mathbf{i})}| <$ CSize. For creating random surfaces that

Table 3: Number of projective symmetries (including the identity)

Ex.	# real symmetries/equivalences	# complex symmetries/equivalences
Σ_1	24	24
Σ_2	8	24
Σ_3	4	24
Σ_4	∞	∞
Σ_5	∞	∞
Σ_6	∞	∞
$\Sigma_1 + \Sigma_2$	0	24
$\Sigma_1 + \Sigma_3$	0	24
$\Sigma_2 + \Sigma_3$	0	24
$\Sigma_4 + \Sigma_5$	0	∞

Table 4: Loglog-plot of the computation time (in sec.) of Gröbner basis for projective equivalences of quadratic rational surfaces with random values.



CSize	1	10	100	1000	10000
symmetry	3.8×10^{-1}	3.2×10^{0}	7.3×10^{0}	1.5×10^{1}	2.5×10^{1}
equivalence	5.0×10^{-1}	3.5×10^{0}	9.8×10^{0}	1.9×10^{1}	3.0×10^{1}
classification	2.4×10^{-1}	6.9×10^{-1}	1.3×10^{0}	2.0×10^{0}	3.0×10^{0}

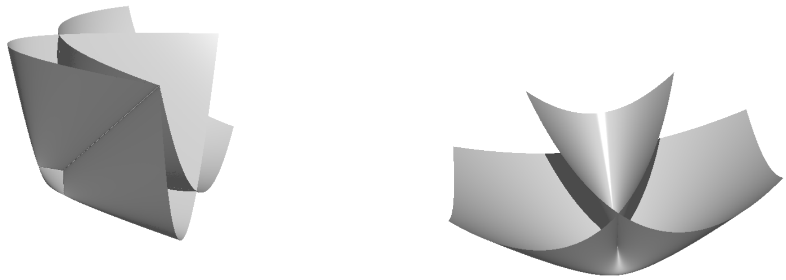
possess equivalences we applied a random reparameterization and transformation on these surfaces. The result of this experiment is given in Table 4.

According to these experimental results, it is faster to classify the surfaces and then to compute the equivalences directly, in particular as the coefficient size is increased. We believe that the reason for the difference in computation time is that for the classification, one of the two input surfaces has fewer non-zero coefficients because it is a pre-computed normal form. The transformation can be obtained by a suitable composition of the transformation into the normal form of the first surface and the inverse of this transformation of the second surface. Computing this inverse is alway possible, as all transformation matrices are regular.

5.2. Affine equivalences of polynomial surfaces

Peters and Reif [19] provided a complete catalogue of all quadratic polynomial surfaces in n-space with respect to affine transformations and they describe a simple method for affinely classifying quadratic polynomial surfaces by investigating affine invariants of the surface. In particular,

- the rank of the linear and the quadratic coefficient matrix,
- the singular set of the surface,
- the set of all possible types of conic sections when intersecting with a hyperplane and
- the set of different types of preimages of hyperplanes which intersect the singular set



3-3-1c 3-3-2b

Figure 2: The general cases of quadratic polynomial surfaces

Table 5: Number of affine classes of quadratic parameterizable surfaces

space dimension	0	1	2	3	4	5	total
# classes	1	5	15	15	5	1	42
# base-point free classes	0	0	0	7	4	1	12

are invariant and together they are sufficient to classify all quadratic polynomial surfaces into 42 equivalence classes. 12 of them are birational and base-point free, see Table 5.

Again we are interested in the number of symmetries for those 12 surfaces that fulfil our assumptions. For the surfaces in three-dimensional space we additionally investigate to which projective class these affine classes belong.

Table 6 summarizes our results and specifies the parameterization of these surfaces. Here we follow the notation of Peters and Reif [19]. Note, that by setting $t_0 = 1$ we obtain the usual polynomial representation, as for all parameterizations the 0-th coordinate equals t_0^2. For surfaces in 3-dimensional space there are two classes (the types 3-3-1c and 3-3-2b) that possess a discrete number of symmetries and these classes are affinely equivalent over the complex numbers. Similarly the types 3-2-1a and 3-2-3 are affinely equivalent over the complex numbers, but they possess a 1-parametric family of symmetries as also do the other types in 3 space. In higher dimensions there is always an infinite number of symmetries, see Table 6.

Similar to the rational case, we are again interested in the generic cases. For space dimensions 3 and 4 we randomly generated 100 instances of coefficients with an absolute value smaller than 100 and investigate which classes are obtained. It turns out that in 3-dimensional space the types 3-3-1c and 3-3-2b are the generic ones, see Table 7. These surfaces possess a finite number of symmetries only, while the remaining types are "more symmetric". For surfaces in 4 space all of the 100 randomly generated surfaces belonged to type 4-3-1b.

Finally we apply our method to randomly generated surfaces and compare the results of directly detecting symmetries and equivalences with classifying the randomly generated surfaces, see Table 8. The computational results in this table refer to the 7 classes of surfaces in 3-dimensional space. Interestingly one sees, that even for coefficients of larger size, the time for solving our system increases moderately and that in the affine case, in contrast to the projective one, classifying the surfaces does not provide an advantage over directly computing the equivalences. However, any of the two approaches can be solved within a few milliseconds.

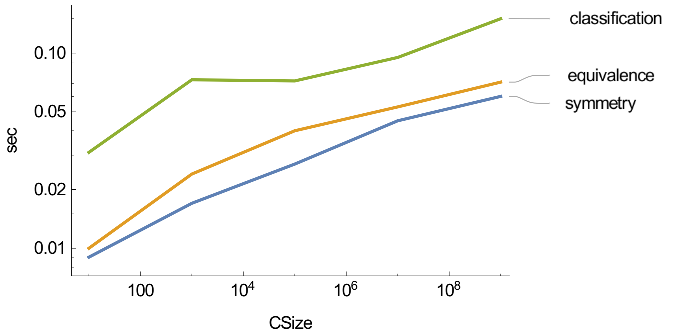
Table 6: Number of the affine symmetries (including the identity)

Type	dim.	parameterization	proj. class.	# real symmetries	# complex symmetries
3-2-1a	3	$(t_0^2, t_1^2, t_2^2, t_0t_1 + t_0t_2)$	Σ_5	∞ (1-par)	∞ (1-par)
3-2-1c	3	$(t_0^2, t_1^2, t_2^2 + t_0t_1, t_0t_2)$	Σ_6	∞ (1-par)	∞ (1-par)
3-2-3	3	$(t_0^2, t_1^2 - t_2^2, t_1t_2, t_0t_1)$	Σ_4	∞ (1-par)	∞ (1-par)
3-3-1b	3	$(t_0^2, t_1^2, t_2^2, t_1t_2 + t_0t_1)$	Σ_5	∞ (1-par)	∞ (1-par)
3-3-1c	3	$(t_0^2, t_1^2, t_2^2, t_0t_1 + t_0t_2 + t_1t_2)$	Σ_1	6	6
3-3-2a	3	$(t_0^2, t_1^2, t_2^2 + t_0t_1, t_1t_2)$	Σ_6	∞ (1-par)	∞ (1-par)
3-3-2b	3	$(t_0^2, t_1^2, t_2^2 + t_0t_1, t_1t_2 - t_0t_2)$	Σ_3	2	6
4-2-1	4	$(t_0^2, t_1^2, t_2^2, t_0t_1, t_0t_2)$		∞ (4-par)	∞ (4-par)
4-2-3	4	$(t_0^2, t_1^2 - t_2^2, t_1t_2, t_0t_1, t_0t_2)$		∞ (4-par)	∞ (4-par)
4-3-1a	4	$(t_0^2, t_1^2, t_2^2, t_1t_2, t_0t_1)$		∞ (3-par)	∞ (3-par)
4-3-1b	4	$(t_0^2, t_1^2 + t_0t_2, t_2^2, t_1t_2, t_0t_1)$		∞ (2-par)	∞ (2-par)
5-3	5	$(t_0^2, t_1^2, t_2^2, t_1t_2, t_0t_1, t_0t_2)$		∞ (6-par)	∞ (6-par)

Table 7: Affine classification of 100 polynomial surfaces with random coefficients $|c_{(k,\mathbf{i})}| \leq 100$

3-2-1a	3-2-1c	3-2-3	3-3-1b	3-3-1c	3-3-2a	3-3-2b	not classified
0	0	0	0	52	0	48	0

Table 8: Computation time (in sec.) of the Gröbner basis for affine equivalences of quadratic polynomial surfaces with random values.



CSize	10^1	10^3	10^5	10^7	10^9
symmetry	9.0×10^{-3}	1.7×10^{-2}	2.7×10^{-2}	4.5×10^{-2}	6.0×10^{-2}
equivalence	1.0×10^{-2}	2.4×10^{-2}	4.0×10^{-2}	5.3×10^{-2}	7.1×10^{-2}
classification	3.1×10^{-2}	7.3×10^{-2}	7.2×10^{-2}	9.5×10^{-2}	1.5×10^{-1}

6. Higher order examples

6.1. Projective and affine equivalences of rational surfaces

As a first example, we consider the surface of the degree 3, which is given by the parameterization

$$\mathbf{p}(\mathbf{t}) = \begin{pmatrix} 30t_0^3 + 3t_0^2t_2 + 3t_1^2t_2 - t_2^3 \\ 30t_0^2t_1 - 10t_1^3 + 30t_1t_2^2 \\ 30t_0^2t_2 + 30t_1^2t_2 - 10t_2^3 \\ 30t_0t_1^2 - 30t_0t_2^2 \end{pmatrix},$$

see Fig. 3 (left), which we obtained by applying a projective transformation to the Enneper surface. For this example, it takes 0.018 seconds to compute the Gröbner basis in Maple. Solving the system confirms that the surface possesses eight real projective symmetries, two of which are even Euclidean symmetries, where one of them is the identity.

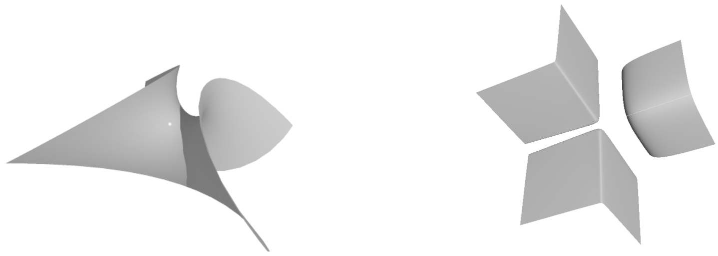


Figure 3: Rational degree 3 surfaces: projectively transformed Enneper (left), surface with 6 symmetries (right)

Table 9: Computation time (in sec.) of Gröbner basis for projective symmetries

Degree \ PZC	80%	70%	60%	50%	40%	0%
2	9.0×10^{-3}	7.5×10^{-2}	2.8×10^{-2}	3.6×10^{0}	6.5×10^{0}	7.6×10^{0}
3	1.6×10^{-2}	6.0×10^{-2}	7.0×10^{-2}	1.6×10^{-1}	1.6×10^{-1}	1.6×10^{-1}
4	2.0×10^{-2}	5.8×10^{-2}	1.5×10^{0}	2.6×10^{0}	2.7×10^{0}	2.7×10^{0}
5	3.4×10^{-2}	3.6×10^{0}	3.4×10^{1}	3.6×10^{1}	3.9×10^{1}	3.6×10^{1}
6	1.1×10^{2}	1.4×10^{2}	1.4×10^{2}	1.5×10^{2}	1.5×10^{2}	1.5×10^{2}

As another example (Fig. 3 right) we consider the surface defined by the parameterization

$$\mathbf{p}(\mathbf{t}) = \begin{pmatrix} t_0t_1t_2 \\ t_0^2t_1 + 3t_0t_1^2 + 3t_1^3 + t_0t_1t_2 + 3t_1^2t_2 + t_1t_2^2 \\ t_0^2t_2 + t_0t_1t_2 + t_1^2t_2 + 3t_0t_2^2 + 3t_1t_2^2 + 3t_2^3 \\ 3t_0^3 + 3t_0^2t_1 + t_0t_1^2 + 3t_0^2t_2 + t_0t_1t_2 + t_0t_2^2 \end{pmatrix}.$$

The computation of the Gröbner basis takes 0.1 seconds. Solving the system reveals that the surface possesses six real symmetries (including the identity) which are all Euclidean ones. Moreover, we used our approach to confirm that the two examples are not projectively equivalent. The computation of the Gröbner basis of the associated polynomial system took 0.071 seconds.

After applying our method to surfaces of degree 2 and 3, we now explore whether it is still feasible for higher degree examples. In addition, we take the possible sparsity of the coefficient matrix into account.

We applied our method to several randomly generated surfaces with a certain percentage of zero coefficients (PZC). The remaining coefficients are integers satisfying $|c_{(k,\mathbf{i})}| < 100$. The computation times of the Gröbner bases are reported in Table 9. As expected, the computation time grows with the degree. We were able to solve the system for dense surfaces of degree 6 within about two minutes. Increased sparsity helps to keep the computational effort low.

6.2. Affine equivalences of polynomial surfaces

It has been observed in Section 5 that the computational effort needed to detect affine equivalences of quadratic polynomial surfaces is substantially smaller than the effort required in the general case. We will now explore how this extends to affine equivalences of higher degree surfaces.

We used a random number generator to create a test suite containing polynomial surfaces up to degree 14, along with linear reparameterizations and affine transformations. All randomly generated coefficients were chosen as integers with an absolute value less than 100. In general these surfaces possess one equivalence (and also only the identity as symmetry).

We were able to detect affine equivalences of degree 14 polynomial surfaces within about two minutes. Table 10 reports the computation times for all degrees up to 14. As to be expected, the computation time depends highly on the input degree, but we can handle higher degrees than in the general case. The detection of equivalences takes longer then the detection of symmetries, but the growth in the computation time with respect to the degree behaves similarly.

Table 10: Computation time (in sec.) of Gröbner basis for affine symmetries and equivalences of polynomial surfaces.

Degree	3	4	5	6	7	8
symm.	5.0×10^{-3}	3.5×10^{-2}	6.8×10^{-2}	4.2×10^{-1}	9.1×10^{-1}	1.9×10^{0}
equiv.	1.0×10^{-2}	6.1×10^{-2}	8.7×10^{-2}	6.6×10^{-1}	1.4×10^{0}	2.9×10^{0}
	9	10	11	12	13	14
symm.	3.6×10^{0}	7.4×10^{0}	1.2×10^{1}	2.0×10^{1}	5.0×10^{1}	8.7×10^{1}
equiv.	5.4×10^{0}	1.1×10^{1}	1.8×10^{1}	3.1×10^{1}	7.6×10^{1}	1.3×10^{2}

7. Conclusion

We observed that two base-point free birational parameterizations of one surface are correlated by a linear reparameterization of the parameter domain, which we identified with the projective plane. This result generalizes the corresponding result for the curve case (Lemma 4.17 of Sendra et al. [21]). Generic rational surfaces of any degree n satisfy the required assumptions on the parameterization (absence of base-point and birationality). It should be noted, however, that they are not satisfied for tensor-product surfaces, where base points are always present.

Based on this result, we propose a method for finding projective and affine equivalences and symmetries of rationally parametrized surfaces. This method creates a polynomial system of equations and reduces the number of unknowns to 9. We solve this system using the Gröbner basis implementation of Maple$^{\text{TM}}$ 2017.

To the best of our knowledge, our method proposed is one of the first approaches capable of finding equivalences and symmetries of rational surfaces. Recently Alcázar and Hermoso [2] investigated involutions of polynomially parametrized surfaces and their method showed a good behaviour also for higher degree surfaces. They solved some special examples up to bidegree (9,11). The considered involutions, however, are special instances of Euclidean symmetries, while our method deals with the more general class of projective transformations. The numerical experiments presented in the paper confirm that projective equivalences and symmetries of randomly generated rational surfaces up to degree 6 and affine equivalences of polynomial surfaces up to degree 14 can be found in less than three minutes on a standard PC.

Two interesting questions for future work arise naturally. First, can we weaken our assumptions, i.e., can we obtain a similar result if the parameterization is not base point free or birational? In particular, the case of tensor-product surfaces should be considered. Second, is it possible to generalize the approach to approximate equivalences? In this paper we assumed that the input data is given by exact values and we applied symbolic methods. The generalization to surfaces defined by floating point numbers is of vital interest.

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