On the Error in Transfinite Interpolation by Low-Rank Functions[⋆]

Nira Dyn[a], Bert Jüttler[b,*], Dominik Mokriš[b]

[a]School of Mathematical Sciences, Tel Aviv University, Israel
[b]Institute of Applied Geometry, Johannes Kepler University, Linz, Austria

Abstract

Given a bivariate function and a finite rectangular grid, we perform transfinite interpolation at all the points on the grid lines. By noting the uniqueness of interpolation by rank-n functions, we prove that the result is identical to the output of Schneider's CA2D algorithm [15]. Furthermore, we use the tensor-product version of bivariate divided differences to derive a new error bound that establishes the same approximation order as the one observed for n-fold transfinite interpolation with blending functions [6].

Keywords: transfinite interpolation, approximation order, low-rank function
2010 MSC: 41A05 65D05

1. Introduction

Transfinite interpolation addresses the task of constructing a function matching given data at a non-denumerable (transfinite) number of points. It was introduced by Gordon and Hall [6], although the particular case of Coons interpolation was proposed before [2]. Applications include mesh generation, geometric modeling, and construction of finite elements accurately capturing boundary conditions. We refer to Sabin's survey [14] for an overview.

Transfinite interpolation has been an active research topic ever since. It was extended to domains that are not of tensor-product type [16]. Kuzmenko and Skorokhodov [12] recently studied transfinite interpolation of functions with bounded Laplacian. The Hermite-Lagrange transfinite interpolation by trigonometric blending functions was also investigated [3], and transfinite mean value interpolation was proposed by Dyken and Floater [4].

Low-rank functions — that is, sums of a low number of separable functions — appear in numerical tensor calculus when using sparse tensor formats for representing multivariate functions [7, 10]. Interpolation by low-rank functions is studied by Schneider [15], and an efficient algorithm for low-rank approximation with bivariate tensor-product splines is proposed [5]. In the context of isogeometric analysis [8], low-rank approximation is successfully applied to address the efficiency problem of matrix assembly [13]. This has motivated us to explore transfinite interpolation by bivariate functions of low rank [9].

The current paper analyzes transfinite interpolation on a finite tensor-product grid by low-rank functions, based on a closed formula in terms of determinants. We focus on uniqueness and error bounds.

[⋆]Supported by the Austrian Science Fund (FWF, NFN S 117).

[*]Corresponding author
Email address: bert.juettler@jku.at (Bert Jüttler)

2. Preliminaries

Consider a bivariate function φ and real values $s_i \in [\underline{s}, \overline{s}]$ and $t_j \in [\underline{t}, \overline{t}]$ with indices $i, j \in \mathbb{N}_0$ that define a tensor-product grid in the domain $\Delta = [\underline{s}, \overline{s}] \times [\underline{t}, \overline{t}] \subset \mathbb{R}^2$. We recall the (non-recursive) definition of the (p, q)-th *divided difference* with respect to the bivariate tensor-product grid [11], which is given by

$$\varphi[s_0, \dots, s_p][t_0, \dots, t_q] = \sum_{i=0}^{p} \frac{1}{\prod\limits_{\substack{k=0,\dots,p \\ k \neq i}} (s_i - s_k)} \underbrace{\sum_{j=0}^{q} \frac{\varphi(s_i, t_j)}{\prod\limits_{\substack{\ell=0,\dots,q \\ \ell \neq j}} (t_j - t_\ell)}}_{= \varphi[s_i][t_0, \dots, t_q]} , \tag{1}$$

under the assumption that $s_i \neq s_j$ and $t_i \neq t_j$ for $i \neq j$. Our first lemma shows how to express the determinant of a sampling matrix $(\varphi(s_i, t_j))_{i,j=0,\dots,m}$ with the help of divided differences.

Lemma 1.

$$\det \big(\varphi(s_i, t_j)\big)_{i,j=0,\dots,m} = \Big(\prod_{0 \leq k < \ell \leq m} (s_\ell - s_k)(t_\ell - t_k)\Big) \det \big(\varphi[s_0, \dots, s_i][t_0, \dots, t_j]\big)_{i,j=0,\dots,m} . \tag{2}$$

Proof. We use Eq. (1) and perform elementary row and column operations (i.e., adding suitable multiples of rows or columns to other ones) for the matrices to confirm the two identities

$$\det \big(\varphi[s_i][t_0, \dots, t_q]\big)_{i,q=0,\dots,m} = \det \left(\frac{\varphi(s_i, t_q)}{\prod\limits_{k=0}^{q-1} (t_q - t_k)} \right)_{i,q=0,\dots,m} \quad \text{and}$$

$$\det \big(\varphi[s_0, \dots, s_p][t_0, \dots, t_q]\big)_{p,q=0,\dots,m} = \det \left(\frac{\varphi[s_p][t_0, \dots, t_q]}{\prod\limits_{k=0}^{p-1} (s_p - s_k)} \right)_{p,q=0,\dots,m} .$$

These combined prove (2), by using the multilinearity of the determinant, and noting that

$$\prod_{0 \leq k < \ell \leq m} (s_\ell - s_k)(t_\ell - t_k) = \prod_{q=0}^{m} \prod_{k=0}^{q-1} (t_q - t_k) \prod_{p=0}^{m} \prod_{k=0}^{p-1} (s_p - s_k) . \qquad \square$$

In the following, we use the abbreviation

$$\varphi^{(k,\ell)}(x, y) = \frac{\partial^k}{\partial x^k} \frac{\partial^\ell}{\partial y^\ell} \varphi(x, y) .$$

Furthermore, for an open set $U \subseteq \mathbb{R}^2$ we will use the symbol $\mathcal{C}^{n,n}(\overline{U})$ to denote the class of functions φ such that all derivatives $\varphi^{(k,\ell)}$, $k, \ell = 0, \dots, n$ are continuous in U and can be continuously extended to $\overline{U}$. We consider a bivariate function $\varphi \in \mathcal{C}^{n,n}(\Delta)$ and recall the bivariate analogue of the mean value theorem for divided differences:

Lemma 2 ([11], Section 11.17)**.** *For any two $k, \ell \in \mathbb{N}_0$ there exists $(\widehat{s}, \widehat{t}) \in \Delta$ such that*

$$\varphi[s_0, \dots, s_k][t_0, \dots, t_\ell] = \frac{\varphi^{(k,\ell)}(\widehat{s}, \widehat{t})}{k!\, \ell!} .$$

From the two previous lemmas, we obtain the following result:

Lemma 3. *There exist* $(\widehat{s}_{ij}, \widehat{t}_{ij}) \in \Delta$, $i, j = 0, \ldots, n$, *such that*

$$\det\big(\varphi(s_i, t_j)\big)_{i,j=0,\ldots,n} = \frac{\prod\limits_{0\le k<\ell\le n} (s_\ell - s_k)(t_\ell - t_k)}{\big(1!\cdots n!\big)^2} \det\big(\varphi^{(i,j)}(\widehat{s}_{ij}, \widehat{t}_{ij})\big)_{i,j=0,\ldots,n} .$$

3. Interpolation by rank-n functions

From now on we assume that mutually different real values $x_i \in [\underline{x}, \overline{x}] = \Omega_x$ and $y_j \in [\underline{y}, \overline{y}] = \Omega_y$ with indices $i, j \in \mathbb{N}$ are given, and we call them *nodes*. Any two nodes $x_i, x_{i'}$ and $y_j, y_{j'}$ with different indices are assumed to be different. Furthermore we define $\Omega = \Omega_x \times \Omega_y$, and we use x_0 and y_0 to denote the variables in order to simplify the notation.

The *rank* of a function $\psi : \Omega \to \mathbb{R}$ is the minimal number r such that there exists a representation of the form

$$\psi(x_0, y_0) = \sum_{k=1}^{r} \gamma_k(x_0)\, \eta_k(y_0) \quad \forall (x_0, y_0) \in \Omega \tag{3}$$

for some functions $\gamma_k : \Omega_x \to \mathbb{R}$ and $\eta_k : \Omega_y \to \mathbb{R}$, $k = 1, \ldots, r$. Note that for such ψ

$$\operatorname{rk}\big(\psi(u_i, v_j)\big)_{i,j=1,\ldots,n} = \operatorname{rk}\Big(\big(\gamma_k(u_i)\big)_{i=1,\ldots,n;k=1,\ldots,r} \cdot \big(\eta_k(v_j)\big)_{k=1,\ldots,r;j=1,\ldots,n}\Big) \le r , \tag{4}$$

for any values $(u_i, v_j) \in \Omega$, $i, j = 1, \ldots, n$, since the rank of a matrix product does not exceed the ranks of its factors.

A function φ is said to be n*-admissible* for some $n \in \mathbb{N}$, with respect to the nodes $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, if the matrix $\big(\varphi(x_i, y_j)\big)_{i,j=1,\ldots,n}$ is non-singular. By (4), any function φ of rank $r < n$ is not n-admissible on Ω.

For any rank-n function ψ, the element $\psi(x_0, y_0)$ of the matrix $(\psi(x_i, y_j))_{i,j=0,\ldots,n}$ – and therefore the function's value at that point – is fully determined by the remaining $(n+1)^2 - 1$ elements if the associated cofactor $\det(\psi(x_i, y_j))_{i,j=1,\ldots,n}$ is non-zero. This is confirmed by the cofactor expansion of the determinant (which, in view of (4), is equal to zero) with respect to the first row (or column). Consequently, *any two n-admissible rank-n functions that take the same values on the tensor-product grid with nodes $x_1, ..., x_n$ and $y_1, ..., y_n$ are identical.*

For an n-admissible function φ, we define the *rank-n approximation operator* with respect to the nodes $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ by

$$(\mathbf{R}_n\varphi)(x_0, y_0) = \frac{-1}{\det\big(\varphi(x_i, y_j)\big)_{i,j=1,\ldots,n}} \det\begin{pmatrix} 0 & \varphi(x_0, y_1) & \cdots & \varphi(x_0, y_n) \\ \varphi(x_1, y_0) & \varphi(x_1, y_1) & \cdots & \varphi(x_1, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(x_n, y_0) & \varphi(x_n, y_1) & \cdots & \varphi(x_n, y_n) \end{pmatrix} . \tag{5}$$

The functions generated by the rank-n approximation operator have rank n or less. This is confirmed by the cofactor expansion of the determinant in the numerator in Eq. (5) with respect to the top row, which produces a decomposition of the form (3) with $\gamma_k(x_0) = \varphi(x_0, y_k)$, with the factors $\eta_k(y_0)$ being the associated cofactors, scaled by the constant that precedes that determinant.

The rank-n approximation operator $\mathbf{R}_n$ performs *transfinite interpolation* on the associated tensor-product grid and is a *projector* onto the set of rank-n functions, as made precise in the following theorem:

Theorem 4. *If φ is n-admissible with respect to the nodes $x_1, \dots, x_n$ and $y_1, \dots, y_n$, then*

$$\varphi(x_0, y_0) - (\mathbf{R}_n\varphi)(x_0, y_0) = \frac{\det\big(\varphi(x_i, y_j)\big)_{i,j=0,\dots,n}}{\det\big(\varphi(x_i, y_j)\big)_{i,j=1,\dots,n}} \ . \tag{6}$$

In particular, $\varphi = \mathbf{R}_n\varphi$ if φ has rank n. Moreover, $\mathbf{R}_n\varphi$ is the unique rank-n function that interpolates φ on the tensor-product grid defined by the nodes, i.e., for all $(x_0, y_0) \in \Omega$

$$\varphi(x_i, y_0) = (\mathbf{R}_n\varphi)(x_i, y_0) \ \textit{and} \ \varphi(x_0, y_j) = (\mathbf{R}_n\varphi)(x_0, y_j), \quad i, j = 1, \dots n.$$

Proof. Using the multilinearity of determinants with respect to the row vectors, we rewrite the numerator $\det\big(\varphi(x_i, y_j)\big)_{i,j=0,\dots,n}$ of the right-hand side in Eq. (6) as

$$\det\begin{pmatrix} \varphi(x_0,y_0) & 0 & \cdots & 0 \\ \varphi(x_1,y_0) & \varphi(x_1,y_1) & \cdots & \varphi(x_1,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(x_n,y_0) & \varphi(x_n,y_1) & \cdots & \varphi(x_n,y_n) \end{pmatrix} + \det\begin{pmatrix} 0 & \varphi(x_0,y_1) & \cdots & \varphi(x_0,y_n) \\ \varphi(x_1,y_0) & \varphi(x_1,y_1) & \cdots & \varphi(x_1,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(x_n,y_0) & \varphi(x_n,y_1) & \cdots & \varphi(x_n,y_n) \end{pmatrix} ,$$

thereby confirming (6) in view of (5). If φ has rank n then (4) implies that the matrix in the numerator of (6) is of rank $\leq n$. Hence its determinant vanishes for all x_0, y_0, and $\mathbf{R}_n\varphi = \varphi$. The interpolation property follows from the fact that the above determinant vanishes if $x_0 = x_i$ or $y_0 = y_j$, $i, j = 1, \dots, n$. Furthermore it allows to invoke the result about the uniqueness of rank-n functions with given values on a tensor-product grid, since the property of n-admissibility is inherited by $\mathbf{R}_n\varphi$ from that of φ. □

The uniqueness of interpolation by rank-n functions implies that $\mathbf{R}_n\varphi$ is the same as the function obtained by Schneider's CA2D algorithm [15], which constructs a function iteratively by performing cross interpolation of the remainder term. It therefore comes as no surprise that the expression for the error derived in Theorem 4 is equivalent to Schneider's Remark 3.3.

4. Error estimates

Now we analyze the L^∞–error of the approximation introduced in the previous section for $\varphi \in \mathcal{C}^{n,n}(\mathbb{R}^2)$. Throughout this section we assume that $\Omega = [0, h]^2$, which implies $0 \leq x_i \leq h$ and $0 \leq y_j \leq h$. The smoothness of φ, together with Lemma 3, implies the following result:

Lemma 5. *If*

$$\det\Big(\varphi^{(i,j)}(0,0)\Big)_{i,j=1,\dots,n} \neq 0 \ , \tag{7}$$

then there exists $h^* > 0$ such that φ is n-admissible for any $h < h^*$.

From now on we will assume that φ is n-admissible with respect to the nodes $x_1, \dots, x_n$ and $y_1, \dots, y_n$. We state the first error bound:

Lemma 6.

$$\|\varphi - \mathbf{R}_n\varphi\|_{L^\infty(\Omega)} \le h^{2n} \sup_{(x_0,y_0)\in\Omega} \left| \frac{\det\left(\varphi[x_0,\dots,x_i][y_0,\dots,y_j]\right)_{i,j=0,\dots,n}}{\det\left(\varphi[x_1,\dots,x_i][y_1,\dots,y_j]\right)_{i,j=1,\dots,n}} \right| \quad . \tag{8}$$

Proof. We use Theorem 4 and Lemma 1 to obtain

$$\|\varphi - \mathbf{R}_n\varphi\|_{L^\infty(\Omega)} = \sup_{(x_0,y_0)\in\Omega} \left| \frac{\prod\limits_{0\le k<\ell\le n} (x_\ell - x_k)(y_\ell - y_k)}{\prod\limits_{1\le k<\ell\le n} (x_\ell - x_k)(y_\ell - y_k)} \right| \left| \frac{\det\left(\varphi[x_0,\dots,x_i][y_0,\dots,y_j]\right)_{i,j=0,\dots,n}}{\det\left(\varphi[x_1,\dots,x_i][y_1,\dots,y_j]\right)_{i,j=1,\dots,n}} \right| ,$$

and complete the proof by noting that

$$\left| \frac{\prod\limits_{0\le k<\ell\le n} (x_\ell - x_k)(y_\ell - y_k)}{\prod\limits_{1\le k<\ell\le n} (x_\ell - x_k)(y_\ell - y_k)} \right| = \left| \prod_{\ell=1}^{n} (x_\ell - x_0)(y_\ell - y_0) \right| \le h^{2n} \quad . \qquad \square$$

We can now state and prove the main result of the paper:

Theorem 7. *If there exists a real number $\overline{h}$ such that*

$$c_\varphi = \frac{\sup\limits_{\widehat{u}_{ij},\widehat{v}_{ij}\in(0,\overline{h})} \left|\det\left(\varphi^{(i,j)}(\widehat{u}_{ij},\widehat{v}_{ij})\right)_{i,j=0,\dots,n}\right|}{\inf\limits_{\breve{u}_{ij},\breve{v}_{ij}\in(0,\overline{h})} \left|\det\left(\varphi^{(i,j)}(\breve{u}_{ij},\breve{v}_{ij})\right)_{i,j=1,\dots,n}\right|} < \infty \quad , \tag{9}$$

then for any $h \in (0, \overline{h})$

$$\|\varphi - \mathbf{R}_n\varphi\|_{L^\infty(\Omega)} \le c_\varphi \frac{h^{2n}}{(n!)^2} \quad . \tag{10}$$

Proof. Consider a particular point $(x_0, y_0) \in \Omega$. We combine Lemma 2 for suitably chosen points $(\widehat{x}_{ij}, \widehat{y}_{ij}) \in \Omega$ and $(\breve{x}_{ij}, \breve{y}_{ij}) \in \Omega$ with assumption (9) and obtain

$$\left| \frac{\det\left(\varphi[x_0,\dots,x_i][y_0,\dots,y_j]\right)_{i,j=0,\dots,n}}{\det\left(\varphi[x_1,\dots,x_i][y_1,\dots,y_j]\right)_{i,j=1,\dots,n}} \right| = \frac{1}{(n!)^2} \left| \frac{\det\left(\varphi^{(i,j)}(\widehat{x}_{ij},\widehat{y}_{ij})\right)_{i,j=0,\dots,n}}{\det\left(\varphi^{(i,j)}(\breve{x}_{ij},\breve{y}_{ij})\right)_{i,j=1,\dots,n}} \right| \le \frac{c_\varphi}{(n!)^2} \quad .$$

This completes the proof of (10) according to Lemma 6, since the above inequality is fulfilled for any point $(x_0, y_0) \in \Omega$. $\square$

Note that assumption (7) of Lemma 5, which guarantees the n–admissibility of φ for domains of size $h < h^*$, also ensures the existence of $c_\varphi < \infty$ for $h < \overline{h}$. Thus, the two assumptions of Theorem 7 are valid if (7) is satisfied.

We briefly compare the error bound to existing results:

– It is proved [6, Theorem 2] that transfinite interpolation with blending functions also gives an approximation error of order h^{2n}.

– Schneider [15, Proposition 2.3] derives an error estimate for $h = 1$, i.e., for functions defined on $\Omega = [0, 1]^2$, which is valid for a particular choice of the nodes ('partial pivoting'). We apply his result to the function $\widetilde{\varphi}(\widetilde{x}_0, \widetilde{y}_0) = \varphi(h\widetilde{x}_0, h\widetilde{y}_0)$ and to the low-rank interpolation operator $\widetilde{\mathbf{R}}_n$ with respect to the nodes $\widetilde{x}_i = x_i/h$ and $\widetilde{y}_i = y_i/h$ and obtain the inequality

$$|(\varphi - \mathbf{R}_n\varphi)(x_0, y_0)| = |(\widetilde{\varphi} - \widetilde{\mathbf{R}}_n\widetilde{\varphi})(\frac{x_0}{h}, \frac{y_0}{h})| \leq \frac{2^n}{n!}\prod_{i=1}^{n}|\frac{x_0}{h} - \widetilde{x}_i| \sup_{\widetilde{u}\in[0,1]} |\widetilde{\varphi}^{(n,0)}(\widetilde{u}, \frac{y_0}{h})|$$
$$= \frac{2^n}{h^n n!}\prod_{i=1}^{n}|x_0 - x_i| \sup_{u\in[0,h]} h^n|\varphi^{(n,0)}(u, y_0)| \leq \frac{2^n}{n!}\, h^n \sup_{u,v\in[0,h]} |\varphi^{(n,0)}(u, v)|$$

for all $x_0, y_0 \in [0, h]$. Consequently, Schneider's result implies approximation order h^n, which is, however, not optimal. This may be caused by the asymmetry with respect to the order of the two variables. However, it should be noted that the error bounds are not directly comparable since Schneider's result applies to a larger class of functions.

5. Closure

We studied transfinite interpolation by bivariate functions of low rank and investigated its uniqueness and approximation power. Future work might address the generalization to the multivariate case (see [1] for cross interpolation of multivariate functions), although the underlying tensor rank is much harder to characterize than matrix rank.

References

[1] Bebendorf, M. [2011]. Adaptive cross approximation of multivariate functions, *Constr. Approx.* **34**: 149–179.

[2] Coons, S. A. [1964]. Surfaces for computer aided design, *Technical Report, MIT* .

[3] Dryanov, D. and Petrov, P. [2012]. Interpolation and L_1-approximation by trigonometric polynomials and blending functions, *J. Approx. Theory* **164**(8): 1049–1064.

[4] Dyken, C. and Floater, M. S. [2009]. Transfinite mean value interpolation, *Comput. Aided Geom. Design* **26**(1): 117–134.

[5] Georgieva, I. and Hofreither, C. [2017]. An algorithm for low-rank approximation of bivariate functions using splines, *J. Comput. Appl. Math.* **310**: 80–91.

[6] Gordon, W. J. and Hall, C. A. [1973]. Construction of curvilinear co-ordinate systems and applications to mesh generation, *Int. J. Numer. Meth. Eng.* **7**(4): 461–477.

[7] Hackbusch, W. [2012]. *Tensor Spaces and Numerical Tensor Calculus*, Springer.

[8] Hughes, T. J. R., Cottrell, J. A. and Bazilevs, Y. [2005]. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, *Comput. Methods Appl. Mech. Engrg.* **194**: 4135–4195.

[9] Jüttler, B. and Mokriš, D. [2017]. Low rank interpolation of boundary spline curves, *Comput. Aided Geom. Design* **55**: 48–68.

[10] Khoromskij, B. N. and Schwab, C. [2011]. Tensor-structured Galerkin approximation of parametric and stochastic elliptic PDEs, *SIAM J. Sci. Comput.* **33**(1): 364–385.

[11] Kunz, K. [1957]. *Numerical Analysis*, McGraw-Hill.

[12] Kuzmenko, D. and Skorokhodov, D. [2016]. Optimization of transfinite interpolation of functions with bounded Laplacian by harmonic splines on box partitions, *J. Approx. Theory* **209**: 44–57.

[13] Mantzaflaris, A., Jüttler, B., Khoromskij, B. and Langer, U. [2015]. Matrix generation in isogeometric analysis by low rank tensor approximation, *in* J.-D. Boissonnat et al. (eds), *Curves and Surfaces*, Lecture Notes in Computer Science, Springer, pp. 321–340.

[14] Sabin, M. [1994]. Transfinite surface interpolation, *Proceedings of the 6th IMA Conference on the Mathematics of Surfaces*, Clarendon Press, pp. 517–534.

[15] Schneider, J. [2010]. Error estimates for two-dimensional cross approximation, *J. Approx. Theory* **162**: 1685–1700.

[16] Várady, T., Rockwood, A. and Salvi, P. [2011]. Transfinite surface interpolation over irregular n-sided domains, *Comput. Aided Design* **43**(11): 1330–1340.