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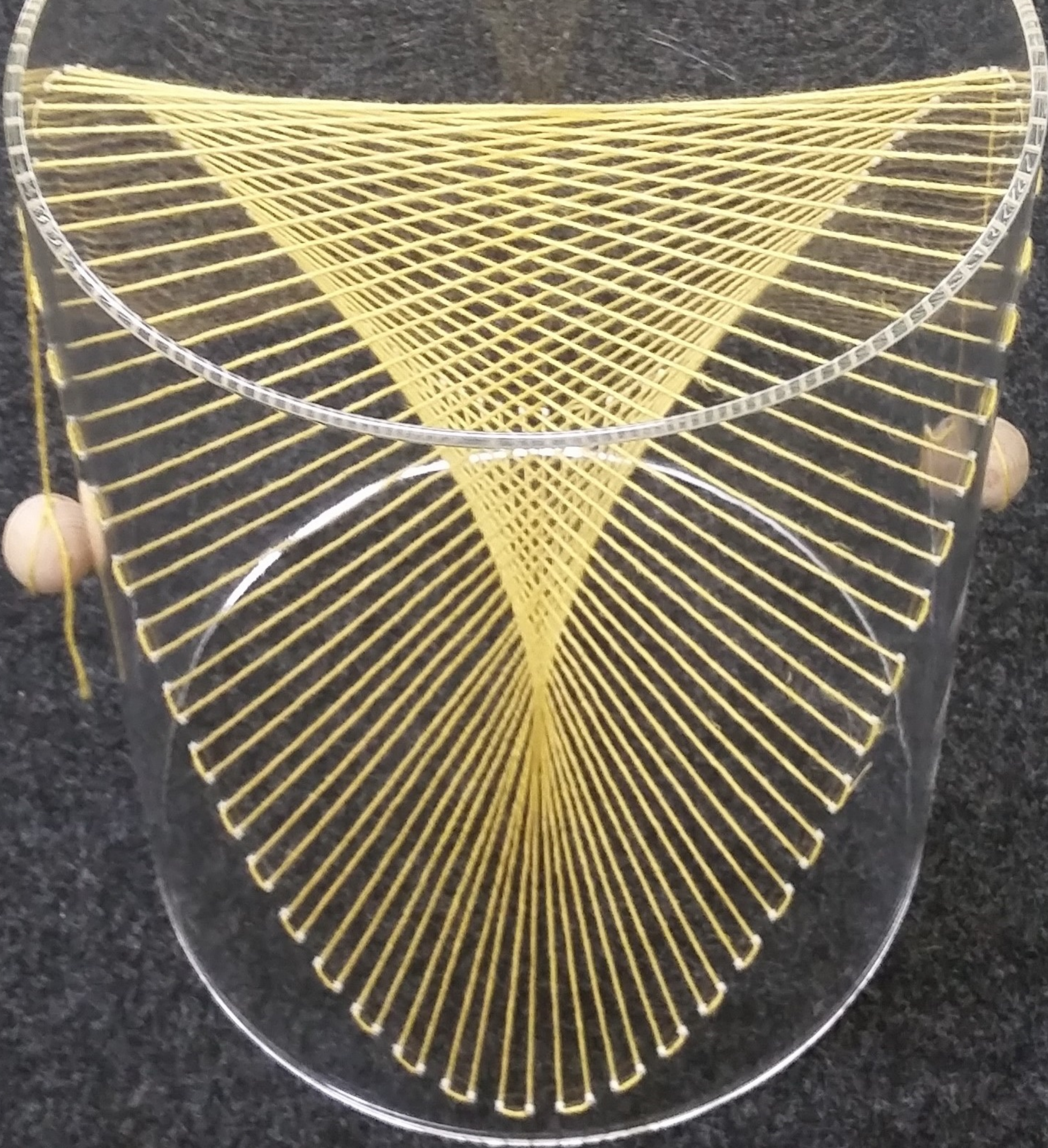

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Apollonian de Casteljau–type Algorithms for Complex Rational Bézier Curves

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Abstract

We describe a new de Casteljau–type algorithm for complex rational Bézier curves. After proving that these curves exhibit the maximal possible circularity, we construct their points via a de Casteljau–type algorithm over complex numbers. Consequently, the line segments that correspond to convex linear combinations in affine spaces are replaced by circular arcs. In difference to the algorithm of Sánchez-Reyes (2009), the construction of all the points is governed by (generically complex) roots of the denominator, using one of them for each level. Moreover, one of the bi-polar coordinates is fixed at each level, independently of the parameter value. A rational curve of the complex degree n admits generically $n!$ distinct de Casteljau–type algorithms, corresponding to the different orderings of the denominator's roots.

Keywords: de Casteljau algorithm, complex rational curve, bi-polar coordinates, Möbius transformation, Farin points

1. Introduction

The survey article of Boehm and Müller (1999), which covers the historical development, concluded with the observation that de Casteljau's algorithm became a fundamental tool in CAGD in the 20 years preceding its publication, and this is even more true today. In this short note, we are interested in generalizations of the original algorithm. Excluding the cases of splines (i.e., piecewise defined curves) and of surfaces, which are beyond the scope of this note, the available results can roughly organized in two categories.

The first one covers generalizations to other ambient spaces, where de Casteljau's algorithm provides a natural generalization of polynomial curves via (e.g.) geodesic interpolation (cf. Popiel and Noakes, 2007; Nava-Yazdani and Polthier, 2013). Among the first works in

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this direction, Shoemake (1985) established generalized Bézier curves on the unit quaternion sphere for computer animation and Park and Ravani (1995) introduced them on Riemannian manifolds and Lie groups for applications in kinematics and motion design, see also the work of Crouch et al. (1999). More recently, Zhang and Noakes (2019) discussed the approximation of cubics in Riemannian spaces via curves defined by de Casteljau's algorithm, and Bogfjellmo et al. (2018) presented a numerical algorithm for creating C^2-smooth curves on manifolds from such curves. An application to the design of ruled surfaces with the help of curves on the Plücker quadric was presented by Nawratil (2019). Applications of generalized Bézier curves on the space of images and on the shape space of planar curves have also been explored (Effland et al., 2015; Samir and Adouani, 2019).

The second category encompasses extensions to other systems of functions. The most classical contribution here is the seminal paper of Farin (1983) about the extension of de Casteljau's algorithm to the case of rational curves based on projective geometry. Extensions to the case of trigonometric curves were presented by Zhu and Han (2015) and Casciola et al. (1998). Even well before that, the (more general) case of Tchebycheff Curves was addressed by Mazure and Pottmann (1996). Numerous other contributions exist, which we do not mention here.

Our work builds on previous contributions to both categories. First, we recall the work of Sánchez-Reyes (2009) on complex rational Bézier curves, which can be associated with the first category. It addresses embeddings of the real line into the projective line over the field of complex numbers. Second, we take into account earlier results obtained by considering the linear factors of the denominators of real rational curves, which have been used to derive a particular version of the de Casteljau algorithm. This idea was first suggested by Han et al. (2014) for a particular form of the denominator. Later it has been extended to general rational curves (Šír and Jüttler, 2015). The resulting algorithm was called a de Casteljau-*type* algorithm since it does not provide the subdivision property, which is available in the classical case.

In the present paper we revisit the framework of complex rational curve and describe a new de Casteljau–type algorithm for complex rational Bézier curves, based on the denominator's linear factors. After proving that these curves exhibit the maximal possible circularity, we construct their points via a de Casteljau–type algorithm over complex numbers. Consequently the usual line segments are replaced by circular arcs. In difference to the algorithm of Sánchez-Reyes (2009), the construction of all the points is governed by (generically complex) roots of the denominator, using one of them per level. Moreover, one of the bi-polar coordinates is fixed at each level, independently of the parameter value. A rational curve of the complex degree n admits generically $n!$ distinct de Casteljau–type algorithms, corresponding to the different orderings of the denominator's roots.

The remainder of the paper is organized as follows. We study the complex rational curves in Section 2 and show that they exhibit the maximal circularity. After recalling bipolar coordinates in Section 3 we design the Apollonian de Casteljau algorithm in Section 4 and we show that the complex rational curves can be constructed with the help of this algorithm in Section 5. Section 6 is devoted to an observation about the Farin points occurring in the complex de Casteljau algorithm of Sánchez-Reyes (2009). We show that they can be obtained solely by intersecting circles. Eventually we conclude the paper.

2. Complex rational Bézier curves

A complex rational Bézier curve (Sánchez-Reyes, 2009) with control points $c_0, \ldots, c_n \in \mathbb{C}$ and associated weights $w_0, \ldots, w_n \in \mathbb{C}$ is defined by the parametric representation

$$p(\xi) = \frac{\sum_{j=0}^{n} \binom{n}{j} \xi^j (1-\xi)^{n-j} w_j c_j}{\sum_{j=0}^{n} \binom{n}{j} \xi^j (1-\xi)^{n-j} w_j} \tag{1}$$

with the real parameter $\xi \in \mathbb{R}$, for some n. We denote the numerator and the denominator of this representation by

$$q(\xi) = \sum_{j=0}^{n} \binom{n}{j} \xi^j (1-\xi)^{n-j} w_j c_j \quad \text{and} \quad r(\xi) = \sum_{j=0}^{n} \binom{n}{j} \xi^j (1-\xi)^{n-j} w_j \ .$$

We assume, without loss of generality, that q and r do not have a common factor (over $\mathbb{C}$). To compute the degree of the curve, we identify $\mathbb{C}$ with $\mathbb{R}^2$, homogenize the parametrization, and cancel common factors that may arise during this process. Assume that $\gcd(r, \overline{r}) =: g$ (a real polynomial), and $s := r/g$. Then we get the real projective parametrization

$$p(\xi) \cong (\Re(q\bar{r}) : \Im(q\bar{r}) : r\bar{r}) = (\Re(q\bar{s}) : \Im(q\bar{s}) : s\bar{s}g), \tag{2}$$

where $\Re(\hat{f})$ and $\Im(\hat{f})$ denote the real and imaginary part of a complex polynomial $\hat{f}$. Then the degree of the curve is equal to the maximum of the degrees of the three polynomials giving the projective parametrization, which is $\max(\deg(q) + \deg(s), 2\deg(s) + \deg(g))$, divided by the degree of the parametrization map in case the parametrization is not proper.

The circularity of a curve is defined as the order of the defining equation at one of the circular points at infinity $(1 : \mathrm{i} : 0)$, $(1 : -\mathrm{i} : 0)$ – for a real curve, the two orders will always coincide. It is relevant for counting intersection points: if C_1 has degree d_1 and circularity γ_1, and C_2 has degree d_2 and circularity γ_2, then the number of intersection points is bounded by $d_1 d_2 - 2\gamma_1\gamma_2$. Circles are simply circular quadrics, and the above formula bounds the number of intersections of two circles by two, while the number of intersections of general quadrics is only bounded by four. Examples of double circular quartics are the lemniscate of Bernoulli, the ovals of Cassini, the cardioid, and the limaçon of Pascal (Cayley, 1874). The difference (degree minus circularity) is invariant under inversive maps (Möbius transformations, including inversion); this is remarkable because the degree is not invariant under inversion. The circularity is bounded by half of the degree. Curves with maximal circularity (such as epi- and hypocycloids) also appear in kinematics, as the orbits of generic rational motions (Li et al., 2016).

For a parametric curve given by $(\hat{f} : \hat{g} : \hat{h})$, the circularity γ is equal to

$$\deg(\gcd(\hat{f} + \mathrm{i}\hat{g}, \hat{h})) = \deg(\gcd(\hat{f} - \mathrm{i}\hat{g}, \hat{h})) \ ,$$

divided by the degree of the parametrization map.

Proposition 1. *The class of complex rational Bézier curve of degree n with* $\gcd(q, r) = 1$ *and* $\deg(r) = n$ *but without proper real factors of the denominator forms the class of rational curves of degree $2n$ with maximum circularity.*

Proof. Let $k \geq 1$ be the degree of the parametrization map (2). Then the circularity of $p(\xi)$ is

$$\frac{\deg(\gcd(\Re(q\bar{s} + \mathrm{i}\Im(q\bar{s}), s\bar{s}g))}{k} = \frac{\deg(\gcd(q\bar{s}, s\bar{s}g))}{k} = \frac{\deg(\bar{s})}{k} = \frac{\deg(s)}{k} .$$

The maximal number for fixed value of the degree is half the degree. This degree is equal to

$$\frac{\max(\deg(q) + \deg(s), 2\deg(s) + \deg(g))}{k} .$$

So we have maximality if and only if $\deg(g) = 0$ and $\deg(q) \leq \deg(s) + \deg(g) = \deg(r)$. $\square$

3. Bipolar coordinates

These coordinates provide the natural setting for representing the intermediate points in the new de Casteljau-type algorithm, which will be described in the next section. Recall (Happel and Brenner, 1983, pp. 516-519) that the *bipolar coordinates* (σ, τ) of the point $z = x + \mathrm{i}y$ in the complex plane with respect to the two foci $F_1 = -1$ and $F_2 = +1$ satisfy the identity

$$z = \mathrm{i}\cot(\frac{\sigma + \mathrm{i}\tau}{2}) . \tag{3}$$

The σ-coordinate evaluates to the natural logarithm of the distance ratio from the point to the two foci, and the τ-coordinate is equal to the angle $\angle(-1, z, 1)$, usually taken in the range $(-\pi, 0) \cup (0, \pi]$. The curves defined by constant values of σ or τ are the *Apollonian circles*, which are mutually orthogonal as they are the image of the Cartesian grid under the holomorphic function (3).

The bipolar coordinates with respect to two general foci F_1, F_2 are obtained by mapping them and z into the standard position via a similarity transformation. For future reference we note the identities

$$\sigma = 2\Re\,\mathrm{arccot}(\mathrm{i}(a - b)) \quad \text{and} \quad \tau = 2\Im\,\mathrm{arccot}(\mathrm{i}(a - b)) ,$$

which are satisfied whenever

$$z = aF_1 + bF_2 \quad \text{and} \quad a + b = 1 .$$

Indeed, they are easily derived from (3) for the standard position, and they extend to general foci since the complex barycentric coordinates

$$a = \frac{F_2 - z}{F_2 - F_1} \quad \text{and} \quad b = \frac{z - F_1}{F_2 - F_1}$$

of z with respect to F_1, F_2 are preserved by similarity transformations.

4. The Apollonian de Casteljau–type algorithm

For given control points $c_0, \ldots, c_n \in \mathbb{C}$ and pairs of weights $(u_k, v_k) \in \mathbb{C}^2 \setminus (0,0)$, (the relation of which to the weights $w_0, \ldots, w_n$ of (1) will be clarified later on, see (7)) with indices $k = 1, \ldots, n$, we generate a curve segment in the complex plane via the following algorithm:

Input: Any $\xi \in [0,1]$
for $j = 0, \ldots, n$ **do**
  $c_j^0 \leftarrow c_j$
end for
for $k = 1, \ldots, n$ **do**
  $a^k \leftarrow \frac{(1-\xi)u_k}{(1-\xi)u_k + \xi v_k}$
  $b^k \leftarrow \frac{\xi v_k}{(1-\xi)u_k + \xi v_k}$
  for $j = 0, \ldots, n-k$ **do**
    $c_j^k \leftarrow a^k c_j^{k-1} + b^k c_{j+1}^{k-1}$
  end for
end for
Return: Point c_0^n on the curve

As a major difference to the classical de Casteljau algorithm, this algorithm uses a different ratio $a^k : b^k$ (which is moreover defined by two complex numbers) for each level k. A example is shown in Figure 1. Starting from four control points (hollow circles), it produces a point in the curve shown in black.

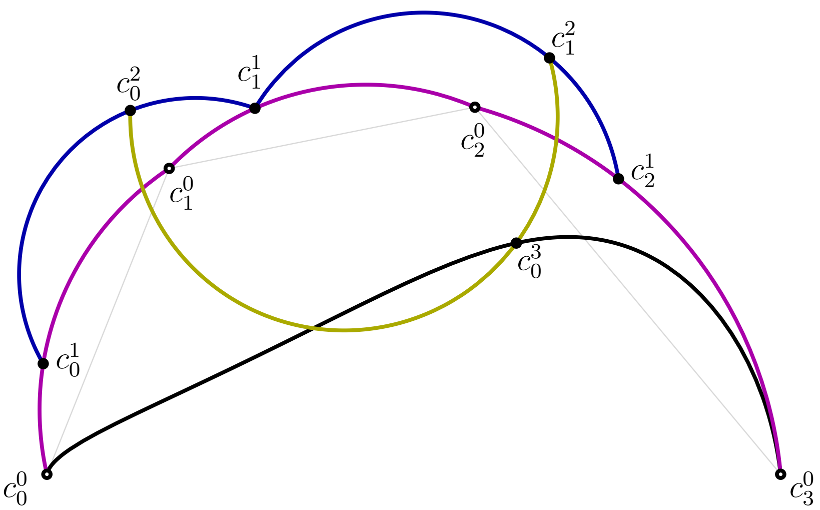

Figure 1: The Apollonian de Casteljau-type algorithm for $n = 3$ and $\xi = 1/2$.

We analyze the linear combination step for some fixed level k. It is governed by the two bilinear complex–valued functions $a^k(\xi)$ and $b^k(\xi)$ that form a partition of unity,

$$a^k(\xi) + b^k(\xi) = 1 \ .$$

Consequently, these two functions specify the barycentric coordinates of the points c_j^{k+1} with respect to c_j^k and c_{j+1}^k.

First we note that – for any given value of ξ – the result depends solely on the ratios

$$q_k = v_k : u_k$$

of the weights. Indeed, the coefficients $a^k(\xi)$ and $b^k(\xi)$ satisfy

$$a^k(\xi) = \frac{(1-\xi)u_k}{(1-\xi)u_k + \xi v_k} = \frac{(1-\xi)}{(1-\xi)+\xi q_k} \ \text{ and } \ b^k(\xi) = \frac{\xi v_k}{(1-\xi)u_k + \xi v_k} = \frac{\xi q_k}{(1-\xi)+\xi q_k} \ . \tag{4}$$

We clarify the relation to the bipolar coordinates:

Proposition 2. *The bipolar coordinates of the points $c_j^{k+1}(\xi)$ with respect to the two foci $c_j^k(\xi)$ and $c_{j+1}^k(\xi)$ take the values*

$$\sigma^k = \arg q_k \quad \textit{and} \quad \tau^k(\xi) = \ln \frac{\xi}{1-\xi} + \ln |q_k| \ .$$

Proof. This follow by using the identity

$$\operatorname{arccot} z = \frac{1}{2\mathrm{i}} \ln \frac{z+\mathrm{i}}{z-\mathrm{i}}$$

to compute the bipolar coordinates (3) with the help of the equation

$$\mathrm{i}\cot(\frac{\sigma^k + \mathrm{i}\tau^k}{2}) = a^k \cdot (-1) + b^k \cdot 1 = \frac{-(1-\xi)+\xi q_k}{(1-\xi)+\xi q_k} \ . \qquad \square$$

Consequently, the Apollonian de Casteljau algorithm proceeds by generating *new points with pre-defined bipolar coordinates* (the same for each level) in each step. Some comments are in order:

- One obtains the classical de Casteljau algorithm if $q_k = 1$ for $k = 1, \ldots, n$.
- All the triangles $\triangle c_j^{k+1}(\xi), c_j^k(\xi), c_{j+1}^k(\xi)$ for $j = 0, \ldots, n-k$ are similar. Consequently, the algorithm is invariant under similarity transformations.
- When choosing pairs of weights with $\arg q_k = \arg v_k : u_k = 0$, all the triangles degenerate into line segments, as $\sigma^k = \pi$. Consequently, the argument of the weight ratios q_k control the deviation of the algorithm from linear rational interpolation. The algorithm is then identical to the the rational de Casteljau–type algorithm of Šír and Jüttler (2015), where the quantities a_k and b_k in that paper are chosen as the norms of u_k and v_k, respectively.

- The coordinate σ^k depends solely on the argument of q_k. It does not on depend on ξ. Consequently, the newly generated points are always located on the *same Apollonian circle* defined by $\sigma = \sigma^k$ with respect to the corresponding bipolar coordinate system.
- The norm of q_k controls the deviation from the chord–length parameterization. If $|q_k| = 1$, then the ratio of the distances from $c_j^{k+1}(\xi)$ to $c_j^k(\xi)$ and $c_{j+1}^k(\xi)$ is equal to the ratio of the distances from ξ to 0 and 1.

For later reference we introduce the *pseudo-Farin points* z_k. These points possess the bipolar coordinates

$$\sigma^k = \arg q_k \quad \text{and} \quad \tau^k(\frac{1}{2}) = \ln |q_k| \tag{5}$$

with respect to the foci 0 and 1, cf. Proposition 2.

5. Application to complex rational Bézier curves

We analyze the curves generated by the algorithm:

Proposition 3. *The Apollonian de Casteljau–type algorithm generates the complex rational Bézier curve* (1) *if*

$$r(\xi) = \prod_{k=1}^{n} [(1-\xi)u_k + \xi v_k] \ . \tag{6}$$

Proof. Mathematical induction allows to prove the identity

$$c_j^k(\xi) = \sum_{\ell=0}^{k} \Big(\sum_{\substack{s_1<\ldots<s_{n-\ell} \\ t_1<\ldots<t_\ell \\ \{s_1,\ldots,s_{n-\ell},t_1,\ldots,t_\ell\}=\{1,\ldots,n\}}} a^{s_1}(\xi)\cdots a^{s_{n-\ell}}(\xi)\cdot b^{t_1}(\xi)\cdots b^{t_\ell}(\xi)\Big)\cdot c_{j+\ell} \ ,$$

where the two sums consider all the decompositions of the level index set $\{1, \ldots, n\}$ into two disjoint subsets $\{s_1, \ldots, s_{n-\ell}\}$ and $\{t_1, \ldots, t_\ell\}$. This is then used to verify that $p(\xi) = c_0^n(\xi)$ takes the rational Bézier form (1) if one chooses the weights

$$w_j = \frac{1}{\binom{n}{j}} \sum_{\substack{s_1<\ldots<s_{n-j} \\ t_1<\ldots<t_j \\ \{s_1,\ldots,s_{n-\ell},t_1,\ldots,t_\ell\}=\{1,\ldots,n\}}} u_{s_1}\cdots u_{s_{n-\ell}}\cdot v_{t_1}\cdots v_{t_\ell} \ , \tag{7}$$

and that the numerator can equivalently be written in the form (6). □

The Apollonian de Casteljau–type algorithm is based on the factorization (6) of the curve's denominator into linear factors. Since we are working in the complex realm, this factorization is always available. As a benefit, it makes it much easier to control the roots of the denominator that must not be real and contained in the parameter domain $[0, 1]$ – no such roots exist if and only if all the the ratios $q_k = v_k : u_k$ are either positive or non-real. This condition is much weaker than requiring non–negative weights for classical (i.e., non-complex) rational Bézier curves!

The previous proposition also implies the following simple observation:

Corollary 4. *The Apollonian de Casteljau–type algorithm gives the same curve for any permutation of the pairs of weights* $(u_1, v_1), \dots, (u_n, v_n)$.

Consequently, there are up to $n!$ de Casteljau–type algorithms that generate the same curve. The maximum is realized for complex rational Bézier curves with only single roots of the denominator. Fig. 2 shows an example.

Note that the results of this section could have been derived also by adapting the approach of Šír and Jüttler (2015). In particular, the weights and the basis functions can be shown to satisfy recurrence relations as noted in Propositions 2 and 3 of the earlier paper.

6. Remarks on Sanchez-Reyes' rational de Casteljau algorithm

Just like their rational counterparts, complex rational Bézier curves can be evaluated with the help of the rational de Casteljau algorithm (see Farin, 1983):

Input: Any $\xi \in [0, 1]$
for $j = 0, \dots, n$ **do**
  $w_j^0 \leftarrow w_j$
  $c_j^0 \leftarrow c_j$
end for
for $k = 1, \dots, n$ **do**
  for $j = 0, \dots, n-k$ **do**
    $w_j^k \leftarrow (1-\xi)\, w_j^{k-1} + \xi\, w_{j+1}^{k-1}$
    $a_j^k \leftarrow (1-\xi)\, \frac{w_j^{k-1}}{w_j^k}$
    $b_j^k \leftarrow \xi\, \frac{w_{j+1}^{k-1}}{w_j^k}$
    $c_j^k \leftarrow a_j^k c_j^{k-1} + b_j^k c_{j+1}^{k-1}$
  end for
end for
Return: Point c_0^n on the curve

It should be noted that one has to generate individual values of the complex barycentric coordinates for each combination of k and j. This is different from the Apollonian de Casteljau-type algorithm, where these coordinates depend on k only.

Sánchez-Reyes (2009) uses the Farin points

$$f_j^k = \frac{w_j^k c_j^k + w_{j+1}^k c_{j+1}^k}{w_j^k + w_{j+1}^k}$$

in order to generalize the geometric interpretation of the rational de Casteljau algorithm to the complex case, noting that the four points c_j^k, c_j^{k+1}, f_j^k and c_{j+1}^k are always concircular (i.e., located on a single circle) and possess the cross ratio $\xi/(1-\xi)$.

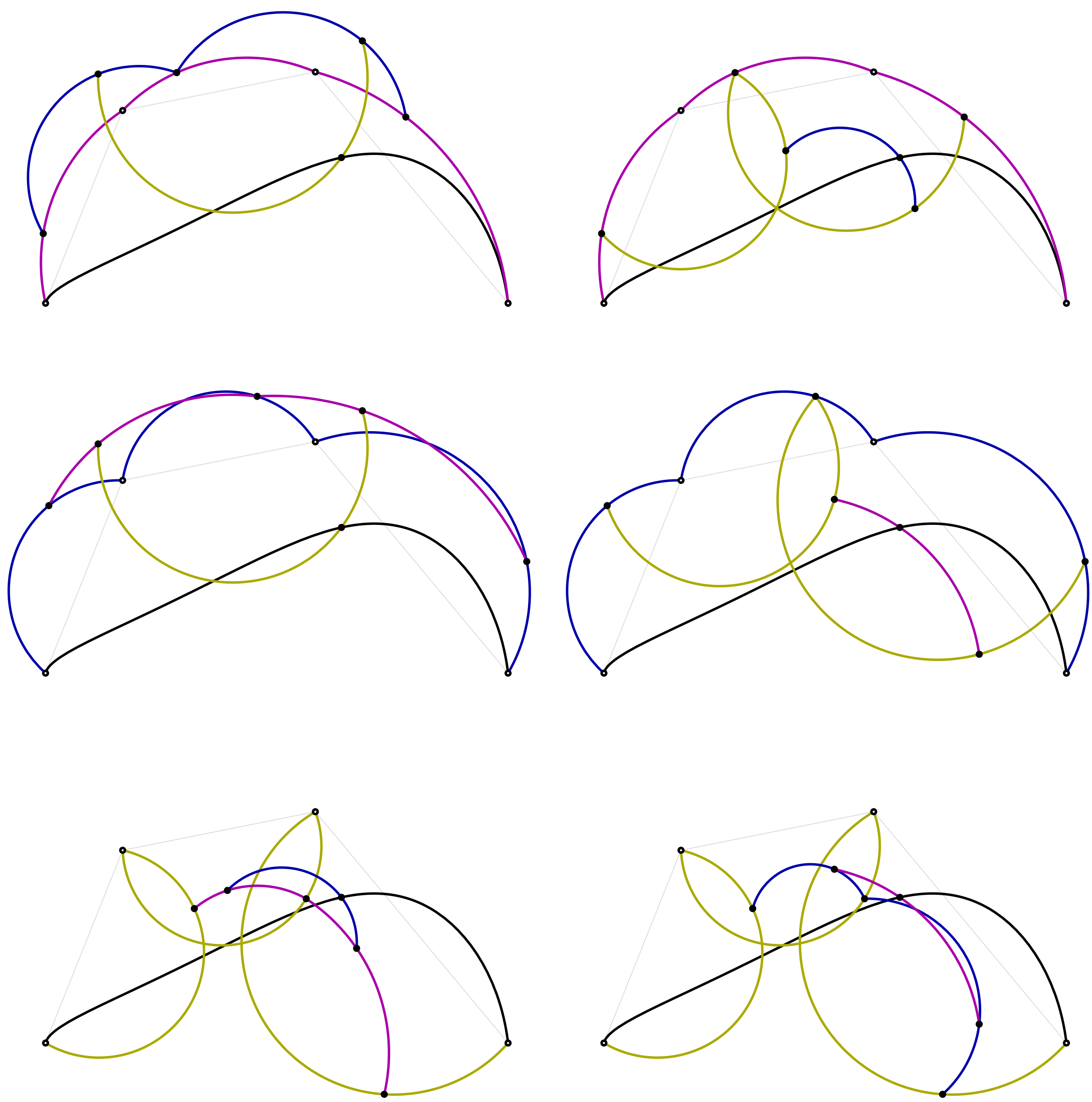

Figure 2: The six permutations of the Apollonian de Casteljau–type algorithm shown in Figure 1.

The level 0 Farin points $f_j^0 = f_j$, $j = 0, \ldots, n-1$, are assumed to be specified by the user, and they can be used as design handles for controlling the shape of the curve. In order to make the generalization complete, one needs to describe how the Farin points of the levels $k \geq 1$ are obtained recursively as well.

In the standard (non-complex) case, Farin (1983) proved that the point f_j^{k+1} is located on the two lines that connect c_j^{k+1} with c_{j+1}^{k+1} and f_j^k with f_{j+1}^k and can thus be found by intersecting them, except in degenerate situations. Consequently, it is possible to realize the rational de Casteljau algorithm by means of a geometric construction that uses solely intersections of lines and cross ratios, thus confirming its invariance under projective mappings.

In the complex case we need to consider the second intersection point g_j^k (in addition to c_{j+1}^k) of the circle through c_j^k, f_j^k and c_{j+1}^k with the circle through c_{j+1}^k, f_{j+1}^k and c_{j+2}^k. We then obtain the following result:

Proposition 5. *The Farin point f_j^{k+1} is located on the two circles through g_j^k that connect c_j^{k+1} with c_{j+1}^{k+1} and f_j^k with f_{j+1}^k.*

Proof. This follows from the arguments in the standard rational case with the help of a Möbius transformation that sends g_j^k to ∞. □

The next level Farin points can thus be found by intersecting certain circles, except in degenerate situations. Consequently, it is possible to realize the complex rational de Casteljau algorithm by means of a geometric construction that uses solely intersections of circles and cross ratios, thus confirming its invariance under Möbius transformations.

The proposition is visualized in Fig 3, where we added the additional intersections to Fig. 3 of Sánchez-Reyes (2009). Note that four circles intersect at each point g_j^k. Sánchez-Reyes (2009) also established the property of *circular precision*, which can now be be re–derived by considering the limit of a sequence of curves where all the control and Farin points of level 0 converge to positions on a given circle.

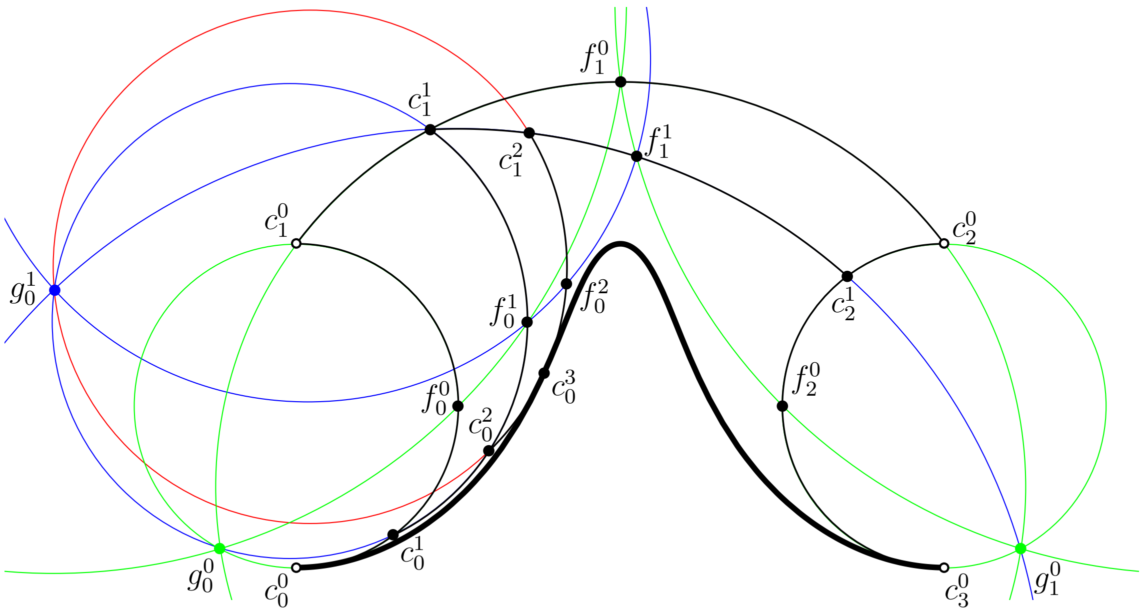

Figure 3: Sanchez-Reyes' rational de Casteljau algorithm

Conclusion

We established a new de Casteljau–type algorithm for complex rational Bézier curves, which can be formulated with the help of bi-polar coordinates. We also analyzed the cir-

	de Casteljau's algorithm	Apollonian de Casteljau-type algorithm	Sánchez-Reyes' (2009) rational de Casteljau algorithm
generates	polynomial Bézier curves	complex rational Bézier curves	
commutes with	affine transformations	similarities	Möbius transformations
subdivision property	yes	no	yes
number of (complex) barycentric coordinates per evaluation	1	n	$\binom{n+1}{2}$
shape handles	control points	plus n pseudo-Farin points z_k	plus n Farin points f_j
absence of points at infinity	always	if and only if $z_k \notin [-\infty, 0] \cup [1, \infty]$ for $k = 1, \ldots, n$	not studied in the complex case, guaranteed by positive weights for real curves (sufficient condition)
precision	linear precision	circular precision for equally spaced control points only	circular precision

Table 1: Comparison of de Casteljau (-type) algorithms.

cularity of rational complex curves and added an observation about the Farin points in the context of the complex rational de Casteljau algorithm of Sánchez-Reyes (2009).

We compare the properties of the two algorithms for complex rational Bézier curves with the standard de Casteljau algorithm in Table 1. Most of them have already been addressed in the previous sections. We would like to stress that our de Casteljau-type algorithms are linked to an alternative geometric representation of complex rational curves, which combines control points c_j, $j = 0, \ldots, n$, and the pseudo-Farin points z_k, $k = 1, \ldots, n$, see (5). The latter ones provide a necessary and sufficient condition for the absence of points at infinity.

In future research we intend to study the analogy of the convex-hull property for the complex de Casteljau algorithms, both in the rational and in the Apollonian case. The generalization to higher dimensions and to curves in general Clifford algebras is also of potential interest.

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