An Algebraic Approach to Curves and Surfaces on the Sphere and on Other Quadrics

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Abstract. An explicit representation for any irreducible rational Bézier curve and Bézier surface patch on the unit sphere is given. The extension to general quadrics (ellipsoids, hyperboloids, paraboloids) is outlined. The construction is based on an algebraic result concerning Pythagorean quadruples in polynomial rings and can be additionally interpreted as a generalized stereographic projection onto the sphere. This projection is shown to be the composition of a hyperbolic projection (a special net projection) with a stereographic projection. The investigation of its properties leads to new results for the biquadratic Bézier patch on the sphere. Further attention is payed to the interpolation of a given point set with a spherical rational curve. The results are extended to rational B-spline curves and tensor product B-spline surfaces.

Keywords. *rational curves and surfaces, unit sphere, Pythagorean quadruple, generalized stereographic projection, net projection, biquadratic Bézier patch, interpolation, B-spline representations, product formula.*

Introduction

Quadric surfaces like spheres, hyperboloids of one or two sheets, elliptic and hyperbolic paraboloids are widely spread in industrial applications. Nevertheless they are not provided by most of Computer-Aided Design systems that are based on parametric representations of curves and surfaces because quadrics cannot be represented by polynomial surface patches in general. A rational surface model avoides this disadvantage: Every quadric can be described as a biquadratic rational Bézier surface. Several authors have dealt with the problem on what condition a biquadratic rational Bézier surface patch describes a part of a quadric and how a specified quadric patch can be obtained as a rational patch ([Boehm & Hansford'91], [Farin & Piper & Worsey'87], [Fink'92], [Geise & Langbecker'90], [Piegl'86], [Sederberg & Anderson'85], [Warren & Lodha'90]).

In this paper an algebraic approach as introduced in [Hoschek & Seemann'92] is used in order to construct Bézier and B-spline curves as well as surfaces on quadrics, especially on the sphere. The constructions are described with help of projective geometry which allows a compact and clear presentation of the results.

1 Some fundamentals of projective geometry

This section presents some fundamentals of projective geometry. For further details, the reader is referred to [Coxeter'64] or [Pedoe'70].
The scene of the following considerations is the projectively closed three-dimensional real Euclidean space $\bar{E}^3$. Its points ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ...) and planes ($\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$, ...) are described by homogeneous coordinate vectors from $\mathbb{R}^4$. The point $\mathbf{a}$ lies on the plane $\hat{\mathbf{b}}$ iff $\langle \mathbf{a}, \hat{\mathbf{b}} \rangle = 0$ holds. (The symbol $\langle .,. \rangle$ denotes the usual inner product of vectors.)
The cartesian coordinate vectors of (finite) points are $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, ... They result from dividing by the 0-th components:

$$\underline{\mathbf{p}} \;=\; \frac{1}{p_0}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \text{where} \quad \mathbf{p} \;=\; \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad . \tag{1}$$

Let $B \neq O$ be a symmetric $(4,4)$-matrix. The set of all points $\mathbf{x}$ satisfying $\mathbf{x}^\top B\,\mathbf{x} = 0$ forms a *quadric*. One of the quadrics in $\bar{E}^3$ is the unit sphere

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0 \quad . \tag{2}$$

The next sections discuss rational parametrizations of the unit sphere. The set of all points of the unit sphere will be denoted by Q.
Plücker's line coordinates are the appropriate tool to handle lines in three-dimensional projective geometry:
Let $\mathbf{x}$ and $\mathbf{y}$ be two distinct points on a given line L. The 16 numbers

$$g_{i.j} \;=\; x_i y_j \;-\; x_j y_i \qquad (i,j = 0,1,2,3) \tag{3}$$

are *Plücker's coordinates of the line* L. These coordinates are homogeneous: If the two points $\mathbf{x}$ and $\mathbf{y}$ are replaced by two arbitrary, but distinct points $\mathbf{x}^*$ and $\mathbf{y}^*$ on the line L, then the coordinates $g^*_{i.j}$ obtained from the latter two points will differ from the coordinates $g_{i.j}$ only by a common factor. The numbers $g_{i.j}$ fulfill the condition

$$g_{i.j} = -g_{j.i} \tag{4}$$

of skew symmetry and Plücker's relation

$$g_{0.1}g_{2.3} \;+\; g_{0.2}g_{3.1} \;+\; g_{0.3}g_{1.2} \;=\; 0 \quad . \tag{5}$$

Thus, only 6 of the 16 numbers $g_{i.j}$ are essential. On the other hand, any 16 numbers $g_{i.j}$ (not all equal to zero) satisfying (4) and (5) are coordinates of a line in $\bar{E}^3$.
The set of all lines whose Plücker's coordinates fulfill a linear equation is called a *linear complex of lines.* These complexes are studied in kinematics: They correspond to screw motions. The lines of the complex connect any point with all points of the normal plane of its trajectory.

2 Algebraic preliminaries

With help of algebraic considerations an explicit representation of parametrized curves and surfaces on the sphere and on other quadrics is derived in this section. This approach has been suggested in [Hoschek & Seemann'92] and [Hoschek'92].
If homogeneous coordinates are used, a rational curve in the projectively closed real Euclidean space $\bar{E}^3$ has the form

$$\mathbf{x}(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \quad , \quad t \in [a, b] \tag{6}$$

where the functions $x_0, \ldots, x_3$ are elements of the polynomial ring $\mathbb{R}[t]$. Analogously triangular and quadrangular surface patches result from (6) by choosing $x_0, \ldots, x_3$ in the quotient ring $\mathbb{R}[u, v, w]/ < u + v + w - 1 >$ or the ring $\mathbb{R}[u, v]$ of bivariate polynomials, respectively. The symbol $[\ldots]$ means the adjunction of an element to a ring and $< \ldots >$ stands for the ideal generated by a ring element.

2.1 Pythagorean quadruples in unique factorization domains

If a rational curve or surface lies on the unit sphere Q, its coordinate functions are a solution of the equation

$$x_0^2 = x_1^2 + x_2^2 + x_3^2 \, . \tag{7}$$

Hence our problem is to find the general solution of (7) in the considered polynomial ring. In number theory it is well known that every solution of the equation

$$x_0^2 = x_1^2 + x_2^2 \tag{8}$$

in a unique factorization domain (UFD) R has the form

$$\begin{array}{rcl} x_0 & = & c(a^2 + b^2) \\ x_1 & = & 2cab \\ x_2 & = & c(a^2 - b^2) \end{array} \quad , \quad a, b, c \in R \tag{9}$$

(see e.g. [Kubota'72]). The solutions x_0, x_1, x_2 are called *Pythagorean triples* in R. An analogous result concerning Pythagorean quadruples which should fulfill (7) was derived by V.A. Lebesgue in the year 1868 and E. Catalan in 1885 (cited in [Dickson'52], pp. 265, 269). A formula yielding every Pythagorean quadruple of integers was given there. When transferring this result to polynomial rings, one needs the

Proposition 2.1 *The polynomial rings $\mathbb{R}[t]$, $\mathbb{R}[u, v]$ and $\mathbb{R}[u, v, w]/ < u + v + w - 1 >$ as well as their complex extensions $\mathbb{C}[t]$, $\mathbb{C}[u, v]$ and $\mathbb{C}[u, v, w]/ < u + v + w - 1 >$ are unique factorization domains.*

Now the following theorem can be formulated:

Theorem 2.2 *If R is one of the polynomial rings $\mathbb{R}[t]$, $\mathbb{R}[u,v,w]/<u+v+w-1>$ or $\mathbb{R}[u,v]$ and the relatively prime polynomials $x_0, x_1, x_2, x_3 \in R$ fulfill the condition*

$$x_0^2 = x_1^2 + x_2^2 + x_3^2 \quad , \tag{10}$$

then x_0, x_1, x_2, x_3 have the form

$$\begin{aligned} x_0 &= p_0^2 + p_1^2 + p_2^2 + p_3^2 \quad \textit{or} \quad x_0 = -p_0^2 - p_1^2 - p_2^2 - p_3^2 \\ x_1 &= 2p_0p_1 - 2p_2p_3 \\ x_2 &= 2p_1p_3 + 2p_0p_2 \\ x_3 &= p_1^2 + p_2^2 - p_0^2 - p_3^2 \end{aligned} \tag{11}$$

with $p_0, p_1, p_2, p_3 \in R$.

Proof. Let C be the complex extension of R, i.e. $\mathbb{C}[t]$, $\mathbb{C}[u,v,w]/<u+v+w-1>$ resp. $\mathbb{C}[u,v]$. Equation (10) can be written in the ring C as

$$xy = z\bar{z} \qquad \text{with} \qquad \mathsf{Im}(x) = \mathsf{Im}(y) = 0 \quad , \tag{12}$$

where $x = \frac{1}{2}(x_0 + x_3)$, $y = \frac{1}{2}(x_0 - x_3)$ and $z = \frac{1}{2}(x_1 + ix_2)$ have been set.
First case: The polynomials x_1 and x_2 are relatively prime in R.
Because C is a UFD it is obvious that there are two factorizations of z — say $z = ac = bd$ — so that $x = a\bar{b}$ holds. Since $\mathsf{Im}(x)$ has to be 0, it follows that

$$\mathsf{Im}(a)\mathsf{Re}(b) - \mathsf{Re}(a)\mathsf{Im}(b) = 0 \quad .$$

The polynomials Re(a) and Im(a) (as well as Re(b) and Im(b)) are relatively prime in R, otherwise x_1 and x_2 would not be relatively prime. Hence $b = r\,a$ with $r \in \mathbb{R}$ can be derived which leads to $x = r\,a\bar{a}$ and analogously $y = s\,c\bar{c}$ with $s \in \mathbb{R}$. Since $rs = 1$ holds and $x = r\,a\bar{a} = \pm(\sqrt{|r|}\,a)(\overline{\sqrt{|r|}\,a})$, without loss of generality $r = s = \pm 1$ can be assumed. For $r = s = 1$ set $a = p_1 + ip_2$ and $c = p_0 + ip_3$ to obtain representation (11) with $x_0 = p_0^2 + \ldots + p_3^2$. For $r = s = -1$ set $a = p_0 + ip_3$ and $c = p_1 + ip_2$ yielding (11) with $x_0 = -p_0^2 - \ldots - p_3^2$.
Second case: The polynomials x_1 and x_2 have a common divisor d.
Assume d to be prime. It is easy to see that the squared divisor d^2 of $z\bar{z} = xy$ is contained in either x or y. Otherwise $x_0, \ldots, x_3$ would have a common factor, in contradiction to the assumption. The factor d^2 of x resp. y can surely be obtained by multiplying a, resp. c with the simple factor d. If d is factorizable, decompose it into prime factors. ■

Now, the above theorem is applied to rational Bézier curves and Bézier surface patches. In section 4 it is generalized to piecewise rational functions, leading to B-spline curve and surface representations.

2.2 Bézier curves and Bézier surface patches on the sphere

A *rational Bézier curve of degree* n is defined as

$$\mathbf{x}(t) = \sum_{i=0}^{n} B_i^n(t)\,\mathbf{b}_i \quad , \tag{13}$$

where $\mathbf{b}_0, \ldots, \mathbf{b}_n \in \mathbb{R}^4$ are the *Bézier control points* in homogeneous coordinates and $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \ldots, n$ are the Bernstein polynomials of degree n. If the coordinate functions of $\mathbf{x}$ are assumed to be relatively prime, i.e. if the curve is irreducible, then there are polynomials $p_0, \ldots, p_3$ which yield the representation (11) of the curve $\mathbf{x}$. Representing these polynomials in the Bernstein basis, one gets

$$\mathbf{p}(t) = \begin{pmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} = \sum_{j=0}^{m} B_j^m(t)\,\mathbf{p}^{(j)} \tag{14}$$

with $\mathbf{p}^{(0)}, \ldots, \mathbf{p}^{(m)}$ as Bézier control points. If (14) is inserted into (11), products of Bernstein polynomials arise which can be computed by the product formula

$$B_i^m\, B_j^n = \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}} B_{i+j}^{m+n} \quad . \tag{15}$$

Of course the resulting curve is of degree $2m$ because $x_0 = p_0^2 + p_1^2 + p_2^2 + p_3^2$ always has this degree. As a corollary, all irreducible curves on the sphere Q are of even degree.
Rational Bézier surface patches can be described in a similar way. A *tensor product Bézier surface patch of degree* (m, n) is defined by

$$\mathbf{x}(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u)\, B_j^n(v)\, \mathbf{b}_{i.j} \tag{16}$$

and a *triangular Bézier surface patch of degree* n is given by

$$\mathbf{x}(u, v, w) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} B_{i.j.k}^n(u, v, w)\, \mathbf{b}_{i.j.k} \tag{17}$$

under the constraint $u + v + w = 1$. The control points are again represented in homogeneous coordinates and in (17) the expression $B_{i.j.k}^n(u, v, w)$ stands for the *generalized Bernstein polynomials*

$$B_{i.j.k}^n(u, v, w) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k \quad .$$

When triangular surface patches are generated using (11), the product formula

$$B_{i_1.j_1.k_1}^m\, B_{i_2.j_2.k_2}^n = \frac{m!\,n!}{(m+n)!} \cdot \frac{(i_1+i_2)!\,(j_1+j_2)!\,(k_1+k_2)!}{i_1!\,j_1!\,k_1!\,i_2!\,j_2!\,k_2!}\, B_{i_1+i_2.j_1+j_2.k_1+k_2}^{m+n}$$

has to be applied. The following corollary is obvious from theorem 2.2:

Corollary 2.3 *Every irreducible rational*

- *Bézier curve of degree* $2n$,
- *tensor product surface of degree* $(2m, 2n)$ *and*
- *triangular surface of degree* $2n$

on the unit sphere Q *can be obtained by formula (11) where* $p_0, \ldots, p_3$ *are*

- *Bernstein polynomials of degree* n,
- *tensor product polynomials of degree* (m, n) *or*
- *generalized Bernstein polynomials over triangular domain of degree* n, *respectively.*

2.3 Extension to other quadrics

The sphere is a special case of the oval quadrics. With help of a projective map every oval quadric can be obtained as image of the sphere. Therefore an approach to find all rational parametrizations on any desired oval quadric is automatically given. The quadric just has to be mapped onto the unit sphere where the representation theorem 2.2 holds and then to be remapped to its original shape.
Another class of non-degenerate quadrics is formed by the doubly ruled quadrics like the hyperboloid of one sheet or the hyperbolic paraboloid. These quadrics are not projectively equivalent to the oval quadrics. But the following theorem holds for the hyperbolic paraboloid with the equation $x_0 x_3 = x_1 x_2$ as a representative for the ruled quadrics:

Theorem 2.4 *If* R *is one of the polynomial rings* $\mathbb{R}[t]$, $\mathbb{R}[u,v,w]/ < u+v+w-1 >$ *or* $\mathbb{R}[u,v]$ *and the polynomials* $x_0, x_1, x_2, x_3 \in R$ *fulfill the condition*

$$x_0 \, x_3 \; = x_1 \, x_2 \; , \tag{18}$$

then there are polynomials $p_0, p_1, p_2, p_3 \in R$ *with*

$$\begin{array}{ll} x_0 = p_0 p_3 & \qquad x_1 = p_1 p_3 \\ x_2 = p_0 p_2 & \qquad x_3 = p_1 p_2 \quad . \end{array} \tag{19}$$

The proof is straightforward and is omitted here. In contrast to an oval quadric a doubly ruled quadric comprises two families of straight lines. These lines are obtained by choosing p_0 and p_1 as constants and p_2, p_3 as linear terms. If p_0 and p_1 are dependent on u only, p_2 and p_3 on v only and all of them are linear, then a bilinear surface patch on the hyperbolic paraboloid is obtained.
The properties of the representation formula for doubly ruled quadrics will be subject of further research. First of all some geometric considerations and results in the case of the sphere are presented:

3 A generalization of stereographic projection

This section contains a geometric interpretation of theorem 2.2. The representation (11) of rational curves and surface patches on the sphere is considered as a generalized stereographic projection $\delta : \bar{E}^3 \to Q$. This projection is shown to be a composition of a stereographic projection with a hyperbolic projection. Some properties of the generalized stereographic projection are discussed. These properties lead to a criterion for biquadratic tensor-product Bézier patches on the sphere interpolating four given boundary curves. Finally, interpolation on the sphere will be shown to be a linear problem.

3.1 The generalized stereographic projection

The representation (11) of rational curves and surface patches on the sphere gives rise to:

Definition. The map $\delta : \bar{E}^3 \to Q$,

$$\delta : \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \mapsto \begin{pmatrix} p_0^2 + p_1^2 + p_2^2 + p_3^2 \\ 2p_0p_1 - 2p_2p_3 \\ 2p_1p_3 + 2p_0p_2 \\ p_1^2 + p_2^2 - p_0^2 - p_3^2 \end{pmatrix} \tag{20}$$

is called the *generalized stereographic projection.*

Now, corollary 2.3 can be formulated with help of this projection: Any irreducible rational Bézier curve of degree $2n$ on the sphere Q can be obtained as the image of a rational Bézier curve of degree n under δ. Analogous assertions hold for rational tensor product Bézier patches and triangular Bézier patches on Q.

3.2 The stereographic projection

The stereographic projection is a standard method for the generation of curves and surfaces on the sphere. Let P denote the plane $x_3 = 0$, i.e. the equator plane of the unit sphere Q, and let $\mathbf{z} = (\,1\,0\,0\,1\,)^\top$. The line connecting the centre $\mathbf{z}$ with an arbitrary point $\mathbf{p} = (\,p_0\,p_1\,p_2\,0\,)^\top$ of P intersects the unit sphere Q in exactly two points (cf. fig.1): in $\mathbf{z}$ and in a second one $\sigma(\mathbf{p})$ where

$$\sigma(\mathbf{p}) = \begin{pmatrix} p_0^2 + p_1^2 + p_2^2 \\ 2p_0p_1 \\ 2p_0p_2 \\ p_1^2 + p_2^2 - p_0^2 \end{pmatrix}. \tag{21}$$

The map $\sigma :\ \mathbf{p} \in P \mapsto \sigma(\mathbf{p}) \in Q$ is called the *stereographic projection* with centre $\mathbf{z}$. The inverse image of a point $\mathbf{q} = (\,q_0\,q_1\,q_2\,q_3\,)^\top \neq \mathbf{z}$ of Q under σ is

$$\sigma^{-1}(\mathbf{q}) = \begin{pmatrix} q_0 - q_3 \\ q_1 \\ q_2 \\ 0 \end{pmatrix}. \tag{22}$$

Figure 1: The stereographic projection

The properties of σ are well known (see e.g.[Coxeter'61]): The stereographic projection and its inversion σ^{-1} preserve circles, i.e. the image of a circle or a line on P under σ is a circle on Q. Additionally, both maps also preserve angles.
Obviously, stereographic projection yields (up to common factors of the homogeneous coordinates) *all* rational parametrizations of the sphere. But, for example, it does not yield all biquadratic Bézier surface patches on the sphere Q as images of bilinear Bézier surface patches on the equator plane P (see e.g.[Geise & Langbecker'90]). The generalized stereographic projection δ avoides this disadvantage of stereographic projection. It will be shown to be a composition of σ with a hyperbolic projection ϑ.

3.3 The hyperbolic projection

The hyperbolic projection ϑ is introduced in order to give a decomposition of the generalized stereographic projection δ :

Definition. The map $\vartheta = \sigma^{-1} \circ \delta : \bar{E}^3 \to P$:

$$\vartheta : \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \mapsto \begin{pmatrix} p_0^2 + p_3^2 \\ p_0p_1 - p_2p_3 \\ p_1p_3 + p_0p_2 \\ 0 \end{pmatrix} \tag{23}$$

is called the *hyperbolic projection.*

The next proposition is obvious from the above definition:

Proposition 3.1 *The generalized stereographic projection δ is composition of the hyperbolic projection ϑ with the stereographic projection σ: $\delta = \sigma \circ \vartheta$.*

Now, some properties of ϑ and δ will be discussed. At first, the inverse image of a point is considered in

Lemma 3.2 *The set of all inverse images of the point* $\mathbf{p} = (\,p_0\, p_1\, p_2\, 0\,)^\top$ *of* P *under* ϑ *(resp. of the point* $\mathbf{q} = (\,q_0\, q_1\, q_2\, q_3\,)^\top$ *of* Q *under* δ *where* $\mathbf{p} = \sigma^{-1}(\mathbf{q})$ *) forms the line*

$$\lambda \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ p_2 \\ -p_1 \\ p_0 \end{pmatrix} \quad (\lambda, \mu \in I\!R)\,. \tag{24}$$

The proof results from straightforward calculations. The lines (24) will be called the *projecting lines* of the hyperbolic projection: Each of them passes through its image under ϑ and it is perpendicular to the line connecting its image under ϑ and the origin. An arbitrary rotation around the axis $(\,1\ 0\ 0\ \lambda\,)^\top$ $(\lambda \in \mathrm{I\!R})$ (i.e. around the z-axis) maps projecting lines to projecting lines. Consider an arbitrary line on the equator plane P through the origin. The projecting lines along this line form a hyperbolic paraboloid (see lemma 3.4).

Theorem 3.3 *The system of all projecting lines of* ϑ *can be generated as the intersection of two linear complexes of lines.*

Proof. The Plücker coordinates of the projecting lines (24) are

$$\begin{array}{lll} g_{0.1} = p_0 p_2 & g_{0.2} = -p_0 p_1 & g_{0.3} = p_0^2 \\ g_{2.3} = p_0 p_2 & g_{3.1} = -p_0 p_1 & g_{1.2} = -p_1^2 - p_2^2 \end{array} \quad . \tag{25}$$

These coordinates satisfy the two linear equations $g_{0.1} - g_{2.3} = 0$ and $g_{0.2} - g_{3.1} = 0$. ■

For example, the two complexes can be chosen corresponding to two null systems describing screw motions with axes $(\,1\ \lambda\ 0\ 0\,)^\top$ and $(\,1\ 0\ \eta\ 0\,)^\top$ $(\lambda, \eta \in \mathrm{I\!R})$ and screw parameters $s_1 = s_2 = 1$.
The intersection of two linear complexes of lines is called a *net of lines* (or linear congruence of lines). A net of lines defines the so called *net projection*: The lines of the net figure as projecting lines. The center plane of the net is the image plane of the projection.
These nets and net projections have been studied in advanced geometry. A net projection can be obtained as parallel projection in a three-dimensional elliptic space (with respect to the Clifford-parallelism) (see e.g. [Wunderlich'36]). Another approach to net projections results from the discussion of screw motions in kinematics (cf. [Tuschel'11], [Bereis & Brauner'57]). *The hyperbolic projection introduced in (23) is a special net projection.* The net of lines formed by the projecting lines (24) is an elliptic one (i.e. all lines of the net pass through two complex skew lines).
The lemmata 3.4, 3.5 and 3.6 are special cases of more general assertions concerning net projections (see e.g. [Brauner'56]).
The next lemma explains the origin of the name *hyperbolic* projection:

Lemma 3.4 *The inverse image of a circle on* P *under* ϑ *(and so of a circle not passing through the centre* $\mathbf{z}$ *on* Q *under* δ*) is a one-sheet-hyperboloid. The inverse image of a line on* P *under* ϑ *(and so of a circle through* $\mathbf{z}$ *on* Q *under* δ*) is a hyperbolic paraboloid.*

Proof. Consider a circle $\mathbf{x}(\varphi) = (\,1 \quad a + rcos\varphi \quad b + rsin\varphi \quad 0\,)$ on P $(\varphi \in [0, 2\pi)$). Its inverse image under ϑ is the ruled surface (cf. lemma 3.2)

$$\mathbf{y}(\varphi, t) = \begin{pmatrix} 1 \\ a + rcos\varphi \\ b + rsin\varphi \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ b + rsin\varphi \\ -a - rcos\varphi \\ 1 \end{pmatrix} \quad (t \in \mathrm{I\!R})\,. \tag{26}$$

Bézout-elimination (see [Salmon 1885]) yields the implicit representation of surface $\mathbf{y}$ as a quadric:

$$\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} a^2+b^2-r^2 & -a & -b & 0 \\ -a & 1 & 0 & -b \\ -b & 0 & 1 & a \\ 0 & -b & a & a^2+b^2-r^2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0 \quad . \tag{27}$$

The matrix in (27) is always nonsingular (for $r \neq 0$) and thus surface $\mathbf{y}$ is a one-sheet-hyperboloid.
Analogous deductions prove the assertion in the case of a line on P. ■

The one-sheet-hyperboloids and hyperbolic paraboloids occuring in lemma 3.4 carry two systems of lines (generators). The first system consists of projecting lines (24) of ϑ. The lines of the second system will be called the *conjugated lines* with respect to the first system. The image of a conjugated line under ϑ is the given circle.
Now, the images of an arbitrary line under ϑ and δ are discussed in

Lemma 3.5 *The image of a given non-projecting line under ϑ is a circle or a line on P. It is a line if and only if the given line intersects the plane P at infinity (i.e. if the given line is parallel to P). Thus the image of a given non-projecting line under δ is a circle on Q.*

Proof. Consider the complex extension of the real projective space. Obviously, the image of a given line under ϑ is a rational quadratic curve, i.e. a conic section on P. This conic section is a circle iff it passes through the *circular points at infinity* $(0\ 1\ i\ 0)$ and $(0\ 1\ -i\ 0)$ of plane P (where i is the imaginary unit). The inverse images of these points under ϑ are the planes $p_3 - i\,p_0 = 0$ and $p_3 + i\,p_0 = 0$. If the given line and P are not parallel, then the line intersects both planes in distinct points, and its image under ϑ is a circle, otherwise it intersects both planes in one point and its image is a line. ■

Finally, the images of a plane under ϑ and δ are discussed:

Lemma 3.6 *Any plane E in $\bar{E}^3$ contains exactly one projecting line L_E. Let two distinct non-projecting lines L_1, L_2 on E be given. Its images under ϑ resp. δ intersect in the two points $\vartheta(L_1 \cap L_2)$ and $\vartheta(L_E)$ resp. in $\delta(L_1 \cap L_2)$ and $\delta(L_E)$. If L_1 and L_2 intersect on L_E, then the tangents of their images under ϑ resp. δ at $\vartheta(L_E)$ resp. $\delta(L_E)$ coincide.*

Thus the image of all lines of a given plane under ϑ resp. δ is the bundle of all circles on P resp. Q through one point. This point corresponds to the projecting line of the given plane. The proof of lemma 3.6 results again from straightforward calculations. Now, a first application of the considerations of this chapter will be given:

3.4 Interpolation with circular arcs

In this section the unique circle interpolating three given points on Q is constructed. The inverse images of the three points on Q under δ are three projecting lines in $\bar{E}^3$. The interpolating circle on Q can be obtained as image of an arbitrary line passing through these three projecting lines (see lemma 3.5).
Let three distinct points on the sphere Q be given: $\mathbf{p}^*$, $\mathbf{q}^*$ and $\mathbf{r}^*$. There is a unique circle

Figure 2: The construction of the conjugated line through the three projecting lines. The image of this line under δ is the circle interpolating $\mathbf{p}^*$, $\mathbf{q}^*$ and $\mathbf{r}^*$.

on Q interpolating these points. Consider the inverse image of this circle under δ. It is a hyperbolic paraboloid or a one-sheet-hyperboloid (see lemma 3.4). The lines $\delta^{-1}(\mathbf{p}^*)$, $\delta^{-1}(\mathbf{q}^*)$ and $\delta^{-1}(\mathbf{r}^*)$ are generating lines of this quadric surface. (see Fig.2)
Each conjugated line of the quadric intersects the three projecting lines and is mapped to the desired circle on Q.
One of the conjugated lines can be obtained as follows: Any plane containing one of the projecting lines, e.g. $\delta^{-1}(\mathbf{r}^*)$, intersects the other two, $\delta^{-1}(\mathbf{p}^*)$ and $\delta^{-1}(\mathbf{q}^*)$, in two points: $\mathbf{b}_0 \in \delta^{-1}(\mathbf{p}^*)$ and $\mathbf{b}_1 \in \delta^{-1}(\mathbf{q}^*)$. Since the line

$$\mathbf{x}(t) = (1-t)\,\mathbf{b}_0 + t\,\mathbf{b}_1 \quad (t \in \mathbb{R} \cup \{\infty\}) \tag{28}$$

passes through each of the three projecting lines, it is a conjugated line of the considered hyperboloid respectively paraboloid. The image of this line under δ is the circle passing through $\mathbf{p}^*$, $\mathbf{q}^*$ and $\mathbf{r}^*$.

Let $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ be preimages of $\mathbf{p}^*$, $\mathbf{q}^*$ and $\mathbf{r}^*$ under δ, i.e. $\delta(\mathbf{p}) = \mathbf{p}^*$, $\delta(\mathbf{q}) = \mathbf{q}^*$ and $\delta(\mathbf{r}) = \mathbf{r}^*$. A preimage of $\mathbf{p}^* = (p_0^*\ p_1^*\ p_2^*\ p_3^*)$ is, for instance, $\mathbf{p} = (p_0^* - p_3^*\ \ p_1^*\ \ p_2^*\ \ 0)$ for $\mathbf{p}^* \neq (1\,0\,0\,1)^\top$ or $\mathbf{p} = (p_1^*\ \ p_0^* + p_3^*\ \ 0\ \ p_2^*)$ for $\mathbf{p}^* \neq (1\ 0\ 0\ -1)^\top$.
Using the notations

$$\mathbf{p} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \mathbf{p}^\perp = \begin{pmatrix} -p_3 \\ p_2 \\ -p_1 \\ p_0 \end{pmatrix} \quad \bar{\mathbf{p}} = \begin{pmatrix} p_2 \\ p_3 \\ -p_0 \\ -p_1 \end{pmatrix}$$

the projecting line $\delta^{-1}(\mathbf{p}^*)$ can be represented by $\lambda\mathbf{p} + \mu\mathbf{p}^\perp$, $\lambda, \mu \in \mathbb{R}$. The points $\mathbf{p}$ and $\mathbf{p}^\perp$ are mapped to the same image point $\mathbf{p}^*$. Obviously, the four coordinate vectors $\mathbf{p}$, $\mathbf{p}^\perp$, $\bar{\mathbf{p}}$ and $\bar{\mathbf{p}}^\perp$ constitute an orthogonal basis of the coordinate space $\mathbb{R}^4$. The following lemma shall be applied to obtain the intersection points $\mathbf{b}_0$ and $\mathbf{b}_1$:

Lemma 3.7 *The intersection point* $\mathbf{s}$ *of the line* $\delta^{-1}(\mathbf{p}^*) = \{\lambda\mathbf{p} + \mu\mathbf{p}^\perp | \, \lambda, \mu \in \mathbb{R}\}$ *and the plane with homogeneous coordinate vector* $\hat{\mathbf{v}}$ *(which does not contain* $\delta^{-1}(\mathbf{p}^*)$ *) is*

$$\mathbf{s} = \langle \hat{\mathbf{v}}, \mathbf{p} \rangle \mathbf{p}^\perp - \langle \hat{\mathbf{v}}, \mathbf{p}^\perp \rangle \mathbf{p} \ .$$

Proof. Clearly $\mathbf{s}$ lies on $\delta^{-1}(\mathbf{p}^*)$ and since $\langle \hat{\mathbf{v}}, \mathbf{s} \rangle = \langle \hat{\mathbf{v}}, \mathbf{p} \rangle \langle \hat{\mathbf{v}}, \mathbf{p}^\perp \rangle - \langle \hat{\mathbf{v}}, \mathbf{p}^\perp \rangle \langle \hat{\mathbf{v}}, \mathbf{p} \rangle = 0$ it lies on the plane $\hat{\mathbf{v}}$, too. ∎

Following the above construction, the plane with coordinate vector $\hat{\mathbf{v}} := \bar{\mathbf{r}}$ is chosen which comprises the projecting line $\delta^{-1}(\mathbf{r}^*)$. The intersection points of that plane with the lines $\delta^{-1}(\mathbf{p}^*)$ and $\delta^{-1}(\mathbf{q}^*)$ are

$$\begin{aligned} \mathbf{b}_0 &= \langle \bar{\mathbf{r}}, \mathbf{p} \rangle \mathbf{p}^\perp - \langle \bar{\mathbf{r}}, \mathbf{p}^\perp \rangle \mathbf{p} \\ \mathbf{b}_1 &= \langle \bar{\mathbf{r}}, \mathbf{q} \rangle \mathbf{q}^\perp - \langle \bar{\mathbf{r}}, \mathbf{q}^\perp \rangle \mathbf{q} \ . \end{aligned}$$

The image curve $\mathbf{x}(t) = \delta(\mathbf{y}(t)) = \delta((1-t)\,\mathbf{b}_0 + t\,\mathbf{b}_1)$, $t \in \mathbb{R} \cup \{\infty\}$ is the circle interpolating $\mathbf{p}^*$, $\mathbf{q}^*$ and $\mathbf{r}^*$. Its Bézier control points can easily be computed.
If the conjugated line $\mathbf{y}(t), t \in \mathbb{R} \cup \{\infty\}$ should pass through the point $\mathbf{p}$ — which one would need when joining another line segment with C^0-continuity — not the plane $\hat{\mathbf{v}} = \bar{\mathbf{r}}$ is chosen, but that of the next

Lemma 3.8 *The plane* $\hat{\mathbf{v}}$*, containing the point* $\mathbf{p}$ *and the line* $\delta^{-1}(\mathbf{r}^*) = \{\lambda\mathbf{r} + \mu\mathbf{r}^\perp | \, \lambda, \mu \in \mathbb{R}\}$ *(which does not contain* $\mathbf{p}$*), has the coordinate vector*

$$\hat{\mathbf{v}} = \langle \mathbf{p}, \bar{\mathbf{r}} \rangle \bar{\mathbf{r}}^\perp - \langle \mathbf{p}, \bar{\mathbf{r}}^\perp \rangle \bar{\mathbf{r}} \ .$$

The proof is similar to that of lemma 3.7.
By analogous deductions a construction of the circle on Q interpolating one point and one point + tangent can be derived from the second part of lemma 3.6. Iterating this construction yields a tangent continuous spherical circular spline through given points.

Figure 3: The condition for biquadratic Bézier patches

3.5 Biquadratic tensor product- and quadratic triangular Bézier patches

In this section, a necessary and sufficient condition for the existence of a rational biquadratic Bézier patch on the sphere Q interpolating four given boundary curves is derived first:
Let four points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ and four circles $C_{1.2}$, $C_{2.3}$, $C_{3.4}$, $C_{4.1} = C_{4.5}$ (where $C_{i.j}$ connects $\mathbf{p}_i$ and $\mathbf{p}_j$) on the sphere Q be given. Additionally let $\mathbf{q}_i$ denote the second intersection point of $C_{i-1.i}$ and $C_{i.i+1}$ $(i = 1, 2, 3, 4)$. (Note that $\mathbf{p}_i = \mathbf{q}_i$ may occur !) A part of the sphere Q is to be described by a rational biquadratic tensor-product Bézier surface patch. The vertices of this patch are the points $\mathbf{p}_i$ and its boundary curves are segments of the circles $C_{i.i+1}$. Under what conditions does such a biquadratic tensor-product Bézier surface patch exist ? The answer is given in

Theorem 3.9 *There exists a biquadratic tensor-product Bézier patch on the sphere Q whose vertices are the points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ and whose boundary curves are segments of the four circles $C_{1.2}$, $C_{2.3}$, $C_{3.4}$, $C_{4.1} = C_{4.5}$ if and only if the two vertices $\mathbf{p}_1$, $\mathbf{p}_3$ and the two second intersection points $\mathbf{q}_2$ and $\mathbf{q}_4$ are coplanar, i.e. if and only if these four points are situated on one circle C on Q (see fig.3).*

Proof. ($\Rightarrow$) Let a biquadratic Bézier patch $\mathbf{y}(u, v)$ on Q be given ($(u, v) \in [0, 1]^2$). There exists a bilinear Bézier patch in $\bar{E}^3$

$$\mathbf{x}(u,v) = B_0^1(u)\,B_0^1(v)\,\mathbf{b}_{0.0} + B_0^1(u)\,B_1^1(v)\,\mathbf{b}_{0.1} + B_1^1(u)\,B_0^1(v)\,\mathbf{b}_{1.0} + B_1^1(u)\,B_1^1(v)\,\mathbf{b}_{1.1} \quad (29)$$

where the given patch $\mathbf{y}(u, v)$ is the image of the bilinear patch $\mathbf{x}(u, v)$ under the generalized stereographic projection δ, i.e. $\mathbf{y}(u, v) = \delta(\mathbf{x}(u, v))$ (see theorem 2.2). The plane spanned by the three points $\mathbf{b}_{0.1}$, $\mathbf{b}_{0.0}$, $\mathbf{b}_{1.0}$ (resp. $\mathbf{b}_{0.1}$, $\mathbf{b}_{1.1}$, $\mathbf{b}_{1.0}$) is denoted by E_2 (resp. by E_4) and the projecting line of this plane (see lemma 3.6) is the line L_2 (resp. the line L_4) (see fig.4). The images of the control points and of the projecting lines are the points

$$\begin{gathered}\delta(\mathbf{b}_{0.1}) = \mathbf{p}_1, \quad \delta(\mathbf{b}_{0.0}) = \mathbf{p}_2, \quad \delta(\mathbf{b}_{1.0}) = \mathbf{p}_3, \quad \delta(\mathbf{b}_{1.1}) = \mathbf{p}_4 \quad \text{and} \\ \delta(L_2) = \mathbf{q}_2, \quad \delta(L_4) = \mathbf{q}_4 \quad .\end{gathered} \quad (30)$$

Figure 4: The proof of theorem 3.9

Obviously the images of the boundary lines of the patch (29) are the boundary curves of patch $\mathbf{y}(u,v)$ and these curves are circles on Q. The three points $\mathbf{b}_{0.1}$, $\mathbf{b}_{0.0}$, $\mathbf{b}_{1.0}$ and the line L_2 are coplanar, thus the boundary curves of patch $\mathbf{y}(u,v)$ connecting $\mathbf{p}_1$ with $\mathbf{p}_2$ and $\mathbf{p}_2$ with $\mathbf{p}_3$ intersect in the vertex $\mathbf{p}_2$ of patch $\mathbf{y}$ and in the point $\mathbf{q}_2$. Analogous deductions show that the remaining two boundary curves between $\mathbf{p}_1$, $\mathbf{p}_3$ and $\mathbf{p}_4$ intersect in the vertex $\mathbf{p}_4$ of patch $\mathbf{y}$ and in the point $\mathbf{q}_4$.
Now consider the line L connecting the two points $\mathbf{b}_{0.1}$ and $\mathbf{b}_{1.0}$. The two planes E_2 and E_4 pass through L. Thus the lines L_2 and L_4 intersect the line L. The image $\delta(L)$ is a circle on Q through the four points $\mathbf{p}_1$, $\mathbf{q}_2$, $\mathbf{p}_3$ and $\mathbf{q}_4$.

($\Leftarrow$) Consider the six lines $\delta^{-1}(\mathbf{p}_1)$, ... , $\delta^{-1}(\mathbf{p}_4)$ and $\delta^{-1}(\mathbf{q}_2)$, $\delta^{-1}(\mathbf{q}_4)$ in $\bar{E}^3$. The four points $\mathbf{p}_1$, $\mathbf{q}_2$, $\mathbf{p}_3$ and $\mathbf{q}_4$ were assumed to be situated on one circle on Q, hence there exists a line L in $\bar{E}^3$ intersecting the four lines $\delta^{-1}(\mathbf{p}_1)$, $\delta^{-1}(\mathbf{q}_2)$, $\delta^{-1}(\mathbf{p}_3)$ and $\delta^{-1}(\mathbf{q}_4)$. (The circle is the image of this line under δ.)
Let $\mathbf{b}_{0.1}$ (resp. $\mathbf{b}_{1.0}$) be the intersection point of the lines L and $\delta^{-1}(\mathbf{p}_1)$ (resp. $\delta^{-1}(\mathbf{p}_3)$). The control points $\mathbf{b}_{0.0}$ and $\mathbf{b}_{1.1}$ are the intersection points of the line $\delta^{-1}(\mathbf{p}_2)$ with the plane spanned by the lines L, $\delta^{-1}(\mathbf{q}_2)$ and of the line $\delta^{-1}(\mathbf{p}_4)$ with the plane spanned by the lines L, $\delta^{-1}(\mathbf{q}_4)$, respectively. Resulting from this construction, the image of the bilinear Bézier patch with the control points $\mathbf{b}_{i.j}$ (cf.(29)) is the required quadratic Bézier patch on the sphere Q. This proves the assertion. ∎

The first part of the theorem is a necessary condition for biquadratic Bézier patches on the sphere. The second part of the proof yields a construction of a biquadratic Bézier patch on Q connecting four given vertices whose boundary curves are segments of given circles. Corresponding to different choices of the boundary curve segments and of the interior sphere segment, the sphere carries 32 patches satisfying these constraints. (Most of them are degenerated.) Only 8 of them can be obtained by the construction presented. (The signs of the weights of the control points $\mathbf{b}_{i.j}$ can be chosen arbitrarily.)

If the second intersection points $\mathbf{q}_i$ are not situated on the boundary curve segments of the patch on the sphere Q and the patch is regular, then the patch can be represented by a biquadratic Bézier patch. This condition is sufficient, but it is not necessary.
The conditions of theorem 3.9 are not fulfilled generally, but:

Theorem 3.10 *There exists always a tensor-product Bézier patch of degree* $(2,4)$ *on the sphere* Q *whose vertices are the points* $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ *and whose boundary curves are segments of the four circles* $C_{1.2}$, $C_{2.3}$, $C_{3.4}$, $C_{4.1} = C_{4.5}$.

Proof. The inverse images of the four boundary circles on the sphere Q under the generalized stereographic projection δ are four one-sheet-hyperboloids resp. hyperbolic paraboloids (lemma 3.4) in $\bar{E}^3$. These four quadric surfaces intersect in projecting lines (24) corresponding to the points $\mathbf{p}_i$ and $\mathbf{q}_i$ $(i = 1,2,3,4)$.
In order to construct the required Bézier patch on the sphere, a rational tensor-product Bézier patch of degree $(1,2)$ has to be found whose boundary curves are situated on the above four quadric surfaces. Two of these four boundaries (on opposite sides of the patch) can be chosen as conjugated lines (generators) of the above quadrics, i.e. as rational linear curves. The remaining two boundary curves must connect the vertices of these conjugated lines. (These vertices are the intersections of the conjugated lines and the projecting lines corresponding to the points $\mathbf{p}_i$.) These boundary curves can be chosen as conic sections, i.e. as rational quadratic curves.
There exists a rational tensor-product-Bézier patch of degree $(1,2)$ interpolating the four boundaries. Applying generalized stereographic projection δ yields the required Bézier patch on the sphere. ■

More detailed constructions of the biquadratic Bézier patch of theorem 3.9 resp. of the Bézier patch of degree $(2,4)$ of theorem 3.10 connecting four given boundary curves will be given in a forthcoming paper [Dietz & Hoschek & Jüttler'93].
In case of rational quadratic triangular Bézier patches, a condition analogous to theorem 3.9 has been developed in e.g.[Sederberg & Anderson'85] [Boehm & Hansford'91]: The boundary curves are again segments of three circles on Q. These circles must intersect in one point.
This condition can also be proved by applying the generalized stereographic projection δ: All quadratic triangular Bézier patches on the sphere can be obtained as images of linear rational triangular Bézier patches in $\bar{E}^3$ under δ. The three control points of this linear patch span a plane in $\bar{E}^3$. The common point of the boundary circles corresponds to the projecting line of this plane.
Consider again the situation of theorem 3.9. Let additionally a circle $C_{1.3}$ on the sphere Q through the two points $\mathbf{p}_1$ and $\mathbf{p}_3$ be given. One of the two segments of the circle $C_{i.j}$ between the points $\mathbf{p}_i$ and $\mathbf{p}_j$ is denoted by $C^*_{i.j}$ $((i,j) = (1,2),(2,3),(3,4),(4,1),(1,3))$. The second intersection point $\mathbf{q}_i$ is assumed not to be situated on the circle segments $C^*_{i-1.i}$ and $C^*_{i.i+1}$ $(i = 1,2,3,4)$. Finally, the circle segments $C^*_{1.3}$, $C^*_{1.2}$ and $C^*_{2.3}$ (resp. $C^*_{1.3}$, $C^*_{4.1}$ and $C^*_{3.4}$) are presumed to be the boundary curves of a regular triangular patch on Q (see fig.5). Then the condition of theorem 3.9 and the above condition for quadratic triangular Bézier patches on the sphere yield the surprising

Corollary 3.11 *(Patchwork-Theorem) The two regular triangular patches on the sphere* Q *connecting the three boundary curve segments* $C^*_{1.3}$, $C^*_{1.2}$ *and* $C^*_{2.3}$ *resp.* $C^*_{1.3}$, $C^*_{4.1}$ *and* $C^*_{3.4}$ *can be described by two quadratic triangular Bézier patches if and only if the regular patch on*

Figure 5: The Patchwork-Theorem

Q connecting the four boundary curve segments $C^*_{1.2}$, $C^*_{2.3}$ $C^*_{3.4}$ and $C^*_{4.1}$ can be represented as a biquadratic tensor product Bézier patch.

The proof is obvious: The circle C of theorem 3.9 corresponds to the circle $C_{1.3}$. The common points of the boundary curves of the triangular Bézier patches are the second intersection points $\mathbf{q}_2$ and $\mathbf{q}_4$ (see fig. 5). (The two quadratic triangular patches are the images of the two linear triangular patches with the control points $\mathbf{b}_{0.0}\,, \mathbf{b}_{0.1}\,, \mathbf{b}_{1.0}$ and $\mathbf{b}_{0.1}\,, \mathbf{b}_{1.0}\,, \mathbf{b}_{1.1}$ in fig. 4 under δ.)
The above corollary holds also in degenerated cases: Figure 6 shows a decomposition of Boehm's biquadratic Bézier patch on the sphere (see [Boehm'93]).
The requirements for the boundary circles of triangular and quadrangular patches can be formulated with help of the vertex angles, too. If a regular triangular Bézier patch is mapped onto a plane by the stereographic projection, its image is a regular triangle — provided that the intersection point of the three boundary circles is chosen as the centre of projection. Because the stereographic map is angle preserving, the three vertex angles of the patch sum up to π. Conversely, if three boundary curves are given which do not intersect and the vertex angles of which sum up to π, a regular Bézier patch with these boundary curves exists.
Applying corollary 3.11 one gets

Corollary 3.12 *Let a regular quadrangular surface patch of the sphere be given. The boundary segments should not pass through the second intersection points $\mathbf{q}_1, \ldots, \mathbf{q}_4$ of the boundary circles and the vertex angles $\varphi_1, \ldots, \varphi_4$ of the corners should lie inside $(0, \pi)$.*
Then a biquadratic tensor product Bézier patch with the given boundary curve segments exists if and only if $\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 2\pi$ holds.

Figure 6: A decomposition of Boehm's patch

Proof. Using the circle $C_{1.3}$ of corollary 3.11, the quadrangular Bézier patch can be subdivided into two triangular patches. The condition $\varphi_2 \in (0, \pi)$ resp. $\varphi_4 \in (0, \pi)$ ensures that the circle segment $C^*_{1.3}$, lying inside the patch, does also not pass through the point $\mathbf{q}_2$ resp. $\mathbf{q}_4$. Therefore, both of the triangular patches are regular and their vertex angles sum up to π each.
The converse direction can be similarly seen. ∎

3.6 Interpolation with rational Bézier curves

Rational spherical curves interpolating given points are constructed in this section. Interpolation on the sphere will be shown to be a *linear* problem.
Let $m+1$ points $\mathbf{p}_i$ with parameters $t_i \in \mathbb{R}$ ($t_0 < t_1 < ... < t_m$) on the sphere Q be given ($i = 0, ..., m$). These points shall be interpolated by a rational Bézier curve $\mathbf{x}(t)$ of degree $2n$ on the sphere Q with $2n+1$ homogeneous control points $\mathbf{b}_j$ ($j = 0, ..., 2n$) (cf.(13)). The generalized stereographic projection will be used in order to construct this curve. The required curve can be obtained as the image of a rational Bézier curve $\mathbf{y}(t)$ of degree n in $\bar{E}^3$ with $n+1$ homogeneous control points $\mathbf{c}_j$ ($j = 0, ..., n$) under δ (see corollary 2.3).
Consider the $m+1$ lines $\delta^{-1}(\mathbf{p}_i)$ in $\bar{E}^3$. For $t = t_i$, the curve $\mathbf{y}(t)$ has to intersect the line $\delta^{-1}(\mathbf{p}_i)$. Then its image $\mathbf{x}(t)$ under δ interpolates the point $\mathbf{p}_i$ on Q.
As already established, the line $\delta^{-1}(\mathbf{p}_i)$ is represented by $\lambda\mathbf{q}_i + \mu\mathbf{q}_i^{\perp}$, $\lambda, \mu \in \mathbb{R}$, where $\mathbf{q}_i$ is a preimage of $\mathbf{p}_i$ under δ, i.e. $\delta(\mathbf{q}_i) = \mathbf{p}_i$. (For the notation $\mathbf{q}^{\perp}$ see section 3.4.)
Therefore the interpolation condition for the curve $\mathbf{y}(t)$ can be written as

$$\mathbf{y}(t_i) = \lambda_i\mathbf{q}_i + \mu_i\mathbf{q}_i^{\perp}\ , \quad i = 0, \dots, 2n \tag{31}$$

with unknown constants $(\lambda_i, \mu_i) \neq (0,0)$. The coordinate vectors $\mathbf{q}_i$, $\mathbf{q}_i^{\perp}$, $\bar{\mathbf{q}}_i$ and $\bar{\mathbf{q}}_i^{\perp}$ form an orthogonal basis of $\mathbb{R}^4$. Hence, the curve $\mathbf{y}(t)$ passes through the line $\delta^{-1}(\mathbf{p}_i)$ iff the two linear equations

$$\langle\bar{\mathbf{q}}_i, \mathbf{y}(t_i)\rangle = 0 \qquad \text{and} \qquad \langle\bar{\mathbf{q}}_i^{\perp}, \mathbf{y}(t_i)\rangle = 0 \tag{32}$$

are fulfilled ($i = 0, ..., m$) and the curve $\mathbf{y}(t)$ does not have a base point at $t = t_i$, i.e. $\mathbf{y}(t_i) \neq \mathbf{0}$. (The equations (32) describe two planes in $\bar{E}^3$. These planes intersect in the projecting line $\delta^{-1}(\mathbf{p}_i)$.)
A homogeneous system of linear equations for the unknown control points $\mathbf{c}_j$, $j = 0, ..., n$, results from

$$\mathbf{y}(t) = \sum_{j=0}^{n} B_j^n(t)\,\mathbf{c}_j \tag{33}$$

and from condition (32). It consists of $2m+2$ equations with $4n+4$ unknowns. Since the two-parametric family of curves $\lambda\mathbf{y}(t) + \mu\mathbf{y}^{\perp}(t)$, $\lambda, \mu \in \mathbb{R}$, leads to the same spherical curve, the number of unknowns can be reduced by 2. Thus, if $m \leq 2n$ holds, then there is at least one nontrivial solution of the homogeneous system of equations.
The system is solved and so a curve $\mathbf{y}(t)$ is found passing through the lines $\delta^{-1}(\mathbf{p}_i)$. Applying generalized stereographic projection δ (see eq.(20)) to the obtained curve $\mathbf{y}(t)$ yields the required interpolating curve $\mathbf{x}(t)$ on the sphere Q. The control points $\mathbf{b}_i$ of the curve $\mathbf{x}(t)$ can be computed with help of product formulas for Bernstein polynomials as mentioned in section 2. Figure 7 shows a quartic rational Bézier curve on the sphere interpolating five given points.
The following theorem deals with the uniqueness of the curve $\mathbf{x}(t)$:

Figure 7: A quartic Bézier curve on the sphere interpolating five given points

Theorem 3.13 *If $2n+1$ points $\mathbf{p}_0, \ldots, \mathbf{p}_{2n}$ on the sphere Q with parameters $t_0, \ldots, t_{2n} \in I\!R$ ($t_i \neq t_j$ for $i \neq j$) are given, then there is a rational spherical curve $\mathbf{x}(t)$ of degree $2n$ satisfying the interpolation problem*

$$\mathbf{x}(t_i) \sim \mathbf{p}_i \ , \quad i = 1, \ldots, 2n \ .$$

The solution is unique in the sense that

$$\mathbf{x}(t) \sim \mathbf{x}^*(t) \ , \quad t \in I\!R$$

holds for all further solutions $\mathbf{x}^*(t)$. ($\sim$ stands for linear dependence and does not exclude $\mathbf{x}(t_i) = \mathbf{0}$!)

Proof. The existence of at least one solution $\mathbf{x}(t)$ is obvious. The second part of the theorem remains to show. Let the coordinate functions of $\mathbf{x}(t)$ be reduced by common divisors and get $x_0 > 0$ by multiplying with -1 if necessary. Then $\mathbf{x}(t)$ is representable as image under δ of a rational curve $\mathbf{y}(t)$ of degree n satisfying equation (31), where $\mathbf{q}_i$ is an arbitrary preimage of $\mathbf{p}_i$. Let $\mathbf{z}(t)$ be a second solution of (31) with $\mathbf{z}(t_i) = \eta_i \mathbf{q}_i + \xi_i \mathbf{q}_i^\perp$, $i = 0, \ldots, 2n$. Then it holds that

$$\langle \mathbf{y}(t_i), \bar{\mathbf{z}}(t_i) \rangle = \langle \lambda_i \mathbf{q}_i + \mu_i \mathbf{q}_i^\perp, \eta_i \bar{\mathbf{q}}_i - \xi_i \bar{\mathbf{q}}_i^\perp \rangle = 0 = \langle \mathbf{y}(t_i), \bar{\mathbf{z}}^\perp(t_i) \rangle \ , \quad i = 0, \ldots, 2n \, .$$

Now, $\langle \mathbf{y}(t), \bar{\mathbf{z}}(t) \rangle$ and $\langle \mathbf{y}(t), \bar{\mathbf{z}}^\perp(t) \rangle$ are polynomials of degree $2n$ at most and have $2n+1$ distinct roots. Thus, they are identical to zero. For every $t^* \in \mathbb{R}$, ν and τ can be found with $\mathbf{y}(t^*) = \nu \mathbf{z}(t^*) + \tau \mathbf{z}^\perp(t^*)$, leading to $\delta(\mathbf{y}(t)) \sim \delta(\mathbf{z}(t))$, $t \in \mathbb{R}$. This proves the assertion. ∎

If the curve $\mathbf{y}(t)$ has a base point at $t = t_i$, then the point $\mathbf{p}_i$ is *inaccessible* in the sense of numerical analysis. The curve $\mathbf{x}(t)$ does not have any infinite points (i.e. poles): The image of an infinite point of the curve $\mathbf{y}(t)$ under the generalized stereographic projection is always finite. If one of the tangents of curve $\mathbf{y}(t)$ is a projecting line of the hyperbolic projection,

then $\mathbf{x}(t) = \delta(\mathbf{y}(t))$ may have a *cusp* at this point.
The method can be extended to interpolation with rational B-Spline curves and surfaces on the sphere, but in this case formulas for the product of B-spline basis functions are needed.

4 B-spline curves and surfaces

We will now extend our constructions to B-Spline curves and tensor-product B-Spline surface patches on quadrics. We get the corresponding curve or surface representation if we insert in (13) or (16) the B-Spline basis functions $N_{ik}(\tau)$ with k as order (degree $k-1$) instead of Bernstein polynomials. The parameter τ may be defined over a knot sequence T. In the interior of T all knots may have multiplicity $l = 1$. Analogously to (15) we need a product formula for the B-Spline basis functions $N_{ik}(\tau)$. For such products we can set

$$N_{ik}(\tau) \cdot N_{jk}(\tau) = \sum_{m=0}^{M} \alpha_m N_{m,2k-1}(\tau) \tag{34}$$

while multiplication of two functions of degree $k-1$ leads to a function of order $2k-1$. Because the B-Spline functions N_{ik} have continuity of order $k-2$ with our assumption for the knot vector, the product of two basis functions must have the same continuity class. Therefore each knot in the knot sequence of $N_{m,2k-1}$ must have multiplicity $l = k$. The coefficients α_m can be determined recursively ([Morken'91], [Vermeulen & Bartels & Heppler'92]) with help of wellknown recursiv definition of B-Spline functions. The factor α_m vanishes if $j \notin \{i, .., i+k-1\}$ otherwise M is determined by

$$\begin{array}{lll} \text{for} \quad j = i & : & M = k(k-3) + 3 \\ \text{for} \quad j = i + \alpha & : & M = k(k - (\alpha+1)) + 1 \qquad \alpha \in \{1, .., k-1\}. \end{array} \tag{35}$$

This formula has an asymmetry: For the product with $i = j$ two additional coefficients appear at the beginning and at the end of the sequence $\{i, .., i + k - 1\}$. The number M is reduced for open spline curves: then we have multiplicity k at the boundaries of the knot sequence and therefore M must be lower. This (very low) continuity C^{k-1} for the B-Spline curves resp. surfaces with basis functions $N_{m,2k-1}$ can be elevated if we choose special sets of control points in the parameter space $\bar{E}^3$:
Consider a linear B-Spline curve in $\bar{E}^3$ with $N+1$ homogeneous control points $\mathbf{p}_i \in \mathbb{R}^4$ $(i = 0, .., N)$. Its image under δ is a quadratic B-Spline curve on the sphere Q. Generally, this curve is only continuous of order 0. The order of continuity can be elevated with help of the second part of lemma 3.6: If the projecting line through $\mathbf{p}_i$ $(0 < i < N)$ and the two control points $\mathbf{p}_{i-1}$ and $\mathbf{p}_{i+1}$ are coplanar, then the image of the linear B-Spline curve under δ is continuous of first order at $\delta(\mathbf{p}_i)$. Using lemma 3.8, this condition can be expressed in terms of coordinate vectors of the control points:

$$\langle \mathbf{p}_{i-1}, \bar{\mathbf{p}}_i \rangle \langle \bar{\mathbf{p}}_i^{\perp}, \mathbf{p}_{i+1} \rangle \quad - \quad \langle \mathbf{p}_{i-1}, \bar{\mathbf{p}}_i^{\perp} \rangle \langle \bar{\mathbf{p}}_i, \mathbf{p}_{i+1} \rangle \quad = \quad 0 \tag{36}$$

If the above equation holds for $i = 1, .., N-1$, then the image of the linear B-Spline curve under δ is a tangent continuous circular spline on the sphere Q.

Conclusion

In this paper, an explicit representation of any irreducible Bézier curve and Bézier surface patch on the unit sphere Q (and on the hyperbolic paraboloid) has been given. This representation has been studied by introducing a generalization of stereographic projection. Discussing this projection, new results concerning the biquadratic Bézier patch on the sphere and methods for interpolation on the sphere have been derived. Rational B-Spline representations have been considered finally.

The extension of this results to other quadrics is straightforward: An arbitrary oval (resp. doubly ruled) quadric surface $\mathbf{x}^\top B\mathbf{x} = 0$ (where B is a nonsingular symmetric (4,4)-matrix) can be obtained as image of the unit sphere (resp. of the hyperbolic paraboloid) under an appropriate projective map

$$\pi : \mathbf{x} \mapsto \pi(\mathbf{x}) = A\mathbf{x}$$

(where A is a nonsingular (4,4)-matrix). Therefore the equations $(A\mathbf{x})^\top B(A\mathbf{x}) = 0$ and $\mathbf{x}^\top Q\mathbf{x} = 0$ (where Q is the matrix corresponding to the unit sphere resp. to the hyperbolic paraboloid (18)) are equivalent. Thus, a representation analogous to (11) resp. (19) for an arbitrary oval (resp. doubly ruled) quadric results as image of that for the unit sphere (resp. of that for the hyperbolic paraboloid) under the projective map π. For example, the hyperboloid of two sheets $x_0^2 + x_1^2 + x_2^2 - x_3^2 = 0$ is obtained from the unit sphere by the projective map

$$\pi_1 : (\, x_0 \; x_1 \; x_2 \; x_3 \,)^\top \mapsto (\, x_1 \; x_2 \; x_3 \; x_0 \,)^\top .$$

The representation analogous to (11) is

$$\begin{aligned} x_0 &= 2p_0p_1 - 2p_2p_3 & \qquad x_1 &= 2p_1p_3 + 2p_0p_2 \\ x_2 &= p_1^2 + p_2^2 - p_0^2 - p_3^2 & \qquad x_3 &= \pm(p_0^2 + p_1^2 + p_2^2 + p_3^2). \end{aligned} \tag{37}$$

The hyperboloid of one sheet $x_0^2 - x_1^2 - x_2^2 + x_3^2 = 0$ results from the hyperbolic paraboloid (18) by the projective map

$$\pi_2 : (\, x_0 \, x_1 \, x_2 \, x_3 \,)^\top \mapsto (\, x_0 + x_3 \quad x_0 - x_3 \quad x_1 + x_2 \quad x_1 - x_2 \,)^\top$$

and the representation analogous to (19) is

$$\begin{aligned} x_0 &= p_0p_3 + p_1p_2 & \qquad x_1 &= p_0p_3 - p_1p_2 \\ x_2 &= p_1p_3 + p_0p_2 & \qquad x_3 &= p_1p_3 - p_0p_2. \end{aligned} \tag{38}$$

The discussion of rational curves and surfaces on quadrics will be continued in a forthcoming paper [Dietz & Hoschek & Jüttler'93]. That paper will present more detailled constructions of the biquadratic Bézier patch and of the Bézier patch of degree (2,4) interpolating given boundary curves.

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