A Geometrical Approach to Curvature Continuous Joints of Rational Curves

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Abstract. Rational Bézier curves are discussed from a projective-geometrical point of view. A projectively invariant Bézier representation of rational curves is presented. Geometric continuity of second and third order is characterized by special projective maps. These maps (certain perspective collineations) preserve curvature properties of a curve at a point. As an application, constructions for geometrically continuous joints of rational curves are derived.

Keywords. Rational Bézier curves, projectively invariant Bézier representation, geometric continuity, rational geometric splines, conic splines, osculating parabola of a conic

Introduction

This paper considers rational curves from a projective-geometrical point of view. First, some fundamentals from projective geometry are presented. Then, in the second part of this paper, a projectively invariant Bézier representation of rational curves is derived. This representation allows to discuss properties of rational curves with help of projective maps.
In recent years, such a representation was found by G. Farin ([Farin'83]). W. Boehm has observed, that similar considerations were made already in 1870 ([Haase 1870]).
The proof of the projective invariance is analogous to that of the affine invariance in the polynomial case: A projectively invariant geometric realization of the de Casteljau-algorithm must be found. This paper presents a realization which is based on Brianchon's theorem.
The third part of this paper contains a characterization of geometric continuity of second and third order by special projective maps (perspective collineations). Geometric continuity of second resp. third order corresponds to a bundle resp. pencil of conics (see [Geise & Nestler'91]). The conics of this bundle or pencil and the given curve(s) have a three- or four-point-contact, respectively. This bundle resp. pencil can be generated by special perspective collineations. So, geometric continuity of second and third order can be characterized by these maps.
A corollary of this characterization is an invariance property of perspective collineations: Certain perspective collineations preserve the curvature of a curve at a point. With help of

this invariance property, a construction of second order continuous joints of rational curves is derived in the fourth part of this paper.
The ratio $\frac{\kappa_1}{\kappa_2}$ of curvatures of two curves which join with tangent continuity at a common point is projectively invariant. This is a result of classical projective differential geometry [Bol'50][Smith 1869]. (An analogous assertion holds for the ratio of torsions.) In [Pottmann'91] and [Goodman'91], this ratio is expressed in terms of cross ratios of control and weight points and so a projectively invariant algorithm for curvature continuous joints of rational curves is found.
In this paper, a *direct geometrical construction* of curvature continuous joints of rational curves is derived with help of the invariance property of perspective collineations. This construction is applied to an interpolating second order continuous conic spline. Furthermore, the construction of the *osculating parabola of a conic* and conditions for the existence of an interpolating third order continuous conic spline are presented.
It is quite surprising, that solely using constructive-geometrical methods leads to such far reaching results.
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1 Fundamentals

In this section, some fundamentals from projective geometry are presented. For further details, the reader is referred to [Coxeter'64] or [Blaschke'54].
The following considerations assume that we are working in the projectively closed real Euclidean plane. Its points (a, b, c, ...) and lines ($\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$, ...) are described by homogeneous coordinate vectors from $\mathbb{R}^3$. The point p lies on the line $\hat{\mathrm{l}}$ iff $\mathrm{p}^\top \hat{\mathrm{l}} = 0$. The cartesian coordinate vectors of (finite) points are $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, ... They result from dividing by the 0-th components:

$$\underline{\mathbf{p}} = \frac{1}{p_0}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad \text{where} \quad \mathrm{p} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} . \tag{1}$$

The symbols $\wedge$ and $\vee$ denote the intersection of lines and the connection of points, respectively. The coordinate vectors of intersection points and of connecting lines can be computed by the usual vector product.
A non-degenerated linear map of $\mathbb{R}^3$

$$\pi : \mathbb{R}^3 \to \mathbb{R}^3 : \quad \begin{cases} \pi(\mathrm{p}) = A\,\mathrm{p} & \text{for points p} \\ \pi(\hat{\mathrm{l}}) = (A^{-1})^\top \hat{\mathrm{l}} & \text{for lines } \hat{\mathrm{l}} \end{cases} \tag{2}$$

(where A is an nonsingular $(3,3)$-matrix) is called a *projective collineation* of the projectively closed real Euclidean plane. It maps points to points and lines to lines.
A projective collineation π is called a *perspective collineation*, if it preserves a line $\hat{\mathbf{a}}$ pointwise (i.e. all points of $\hat{\mathbf{a}}$ are fixed points of π), and a point z linewise (i.e. all lines through z are fixed lines of π). The line $\hat{\mathbf{a}}$ is the *axis*, the point z is the *centre* of π. (If $\mathrm{z}^\top \hat{\mathbf{a}} \neq 0$, then the matrix A of a perspective collineation has exactly two real eigenvalues: a single and a double one, where all eigenspaces are non-degenerated.)
A perspective collineation is uniquely determined by the axis $\hat{\mathbf{a}}$, the centre z and one pair

Figure 1: A perspective collineation

$(\mathrm{p}, \pi(\mathrm{p}))$ consisting of a point p and its image under π (where the line $\mathrm{p} \vee \pi(\mathrm{p})$ passes through z and the axis as well as the centre are not incident with p and $\pi(\mathrm{p})$). The image $\pi(\mathrm{x})$ of an arbitrary point x under π can be constructed by (cf. fig.1)

$$\begin{array}{lll} 1.) & \mathrm{q} & := \quad \hat{\mathbf{a}} \wedge (\mathrm{x} \vee \mathrm{p}) \\ 2.) & \pi(\mathrm{x}) & := \quad (\mathrm{z} \vee \mathrm{x}) \wedge (\mathrm{q} \vee \pi(\mathrm{p})) \end{array} . \tag{3}$$

Let $\mathrm{a} = \alpha_1\,\mathrm{p} + \alpha_2\,\mathrm{q}$, $\mathrm{b} = \beta_1\,\mathrm{p} + \beta_2\,\mathrm{q}$, $\mathrm{f} = \varphi_1\,\mathrm{p} + \varphi_2\,\mathrm{q}$ and $\mathrm{x} = \xi_1\,\mathrm{p} + \xi_2\,\mathrm{q}$ be four points on a line $\mathrm{p} \vee \mathrm{q}$ $(\alpha_1, .., \xi_2 \in \mathbb{R})$. The real number

$$cr(\mathrm{a}, \mathrm{b}, \mathrm{f}, \mathrm{x}) := \frac{det\begin{pmatrix} \alpha_1 & \xi_1 \\ \alpha_2 & \xi_2 \end{pmatrix} \cdot det\begin{pmatrix} \varphi_1 & \beta_1 \\ \varphi_2 & \beta_2 \end{pmatrix}}{det\begin{pmatrix} \varphi_1 & \xi_1 \\ \varphi_2 & \xi_2 \end{pmatrix} \cdot det\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}} \tag{4}$$

is called the *cross ratio* of the four points. It is invariant under projective maps. If all points are finite, it can be expressed in terms of signed (oriented) Euclidean distances $dist(.,.)$:

$$cr(\mathrm{a}, \mathrm{b}, \mathrm{f}, \mathrm{x}) := \frac{dist(\mathrm{a}, \mathrm{x}) \cdot dist(\mathrm{f}, \mathrm{b})}{dist(\mathrm{f}, \mathrm{x}) \cdot dist(\mathrm{a}, \mathrm{b})}. \tag{5}$$

The four points a, b, f, x are said to be *in harmonic position* if their cross ratio is equal to 0.5, -1 or 2.
If the last point x runs through $\hat{\mathrm{l}}$, equation (4) defines a *projective scale on the line* $\hat{\mathrm{l}}$, i.e. a bijective map $\hat{\mathrm{l}} \to \mathbb{R} \cup \{\infty\}$.
Let B be a symmetric nonsingular $(3,3)$-matrix. The set of all points x satisfying $\mathrm{x}^\top B\,\mathrm{x} = 0$ forms a *conic*.
A conic is uniquely determined by five of its tangents. Six tangents $\hat{\mathrm{l}}_i$ $(i = 1, .., 6)$ of a conic are connected by Brianchon's theorem: The three lines $(\hat{\mathrm{l}}_i \wedge \hat{\mathrm{l}}_j) \vee (\hat{\mathrm{l}}_k \wedge \hat{\mathrm{l}}_l)$ (where $(i,j,k,l) \in \{(1,2,4,5),(2,3,5,6),(3,4,6,1)\}$) intersect in one point, the so-called Brianchon's point.
Let two projective scales on two lines $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ be given. The system of lines connecting corresponding points of both scales (i.e. points which correspond to the same real number) envelops a conic or all lines pass through one point (Steiner's generation of conics). This fact yields the parametric representation of a conic as a rational curve of degree two.

2 Rational curves and their Bézier polygon

A projectively invariant Bézier representation of rational curves is presented in this section. As a first application, linear-fractional parameter transformations of these curves are discussed.

Figure 2: The Bézier polygon of a rational curve

2.1 Interior and exterior weight points

Let

$$\mathrm{x}(t) = \sum_{i=0}^{n} B_i^n(t)\,\mathbf{b}_i \quad t \in [0,1] \tag{6}$$

(with the Bernstein polynomials $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$) be the homogeneous coordinates of a given rational Bézier curve of degree n. Usually, the control points $\mathbf{b}_i \in \mathbb{R}^3$ are written in the inhomogeneous form

$$\mathbf{b}_i = w_i \cdot \begin{pmatrix} 1 \\ \underline{\mathbf{b}}_i \end{pmatrix} \tag{7}$$

with *weights* $w_i \in \mathbb{R}$. Obviously, this description excludes points at infinity. In addition to the control points, *interior weight points* $\mathrm{w}_{i.i+1} = \mathbf{b}_i + \mathbf{b}_{i+1}$ and *exterior weight points* $\mathrm{f}_{i.i+1} = \mathbf{b}_i - \mathbf{b}_{i+1}$ are introduced ($i = 0,..,n-1$). The control and weight points form the *Bézier polygon* of the given curve (see fig.2). The interior weight points have been introduced by Farin [Farin'83]. The point $\mathrm{w}_{i.i+1}$ divides the line segment $\mathbf{b}_i \vee \mathbf{b}_{i+1}$ of the Bézier polygon by the ratio $w_{i+1} : w_i$. The four points $\mathbf{b}_i, \mathbf{b}_{i+1}, \mathrm{w}_{i.i+1}$ and $\mathrm{f}_{i.i+1}$ are in harmonic position. They define a projective scale on the line $\mathbf{b}_i \vee \mathbf{b}_{i+1}$:

$$\begin{array}{ccc} \mathbf{b}_i \vee \mathbf{b}_{i+1} & \rightarrow & \mathbb{R} \cup \{\infty\} \\ \mathrm{p} & \mapsto & cr(\mathbf{b}_i, \mathbf{b}_{i+1}, \mathrm{f}_{i.i+1}, \mathrm{p}) \end{array} \quad . \tag{8}$$

With respect to this scale, the four points $\mathbf{b}_i$, $\mathrm{w}_{i.i+1}$, $\mathbf{b}_{i+1}$ and $\mathrm{f}_{i.i+1}$ correspond to 0, 0.5, 1 and ∞, respectively.

2.2 Geometric realizations of the de Casteljau-algorithm

With help of the projective scales on the lines of the Bézier polygon, geometric realizations of the de Casteljau-algorithm can be derived. Farin has developed such a realization using constant cross ratios [Farin'83]. Another one is based on Brianchon's theorem (Similar considerations can already be found in [Haase 1870] ! This paper discusses the existence of a rational parametric representation of an algebraic curve. Then the parameter of this representation is shown to have a projective-geometrical meaning. A de Casteljau-like construction based on Brianchon's theorem is derived finally.):
The basic step of the de Casteljau-algorithm is replaced by the *doublestep*:

$$\begin{array}{cc} \mathrm{b}_i^j & \\ & \mathrm{b}_i^{j+1} \\ \mathrm{b}_{i+1}^j & \\ & \mathrm{b}_{i+1}^{j+1} \\ \mathrm{b}_{i+2}^j & \end{array} \mapsto \mathrm{b}_i^{j+2} = B_0^2(t)\,\mathrm{b}_i^j + B_1^2(t)\,\mathrm{b}_{i+1}^j + B_2^2(t)\,\mathrm{b}_{i+2}^j \quad . \tag{9}$$

Rational curves of degree two are conics. A *tangent element* of a curve is "tangent *plus* point of contact". The conic (9) is determined by the two tangent elements " $(\mathrm{b}_i^j \vee \mathrm{b}_i^{j+1})$ *plus* b_i^j "

Figure 3: The doublestep-construction

Figure 4: A geometric realization of the de Casteljau-algorithm

and " $(b^j_{i+2} \vee b^{j+1}_{i+1})$ *plus* b^j_{i+2} " and by the tangent $b^{j+1}_i \vee b^{j+1}_{i+1}$. The point b^{j+2}_i is the point of contact of the conic and its tangent $b^{j+1}_i \vee b^{j+1}_{i+1}$. It can be constructed by applying Brianchon's theorem to three tangent elements. These tangent elements are considered as *double tangents*. The points of contact figure as intersection points of these double tangents (fig. 3):

$$\begin{array}{llcl} 1.) & \mathrm{p} & = & (b^{j+1}_i \vee b^j_{i+2}) \wedge (b^j_i \vee b^{j+1}_{i+1}) \\ 2.) & b^{j+2}_i & = & (b^j_{i+1} \vee \mathrm{b}) \wedge (b^{j+1}_i \vee b^{j+1}_{i+1}) \end{array} \quad (10)$$

The point p is Brianchon's point.
With help of this construction, the de Casteljau-algorithm can be realized geometrically: Let the parameter $t \in [0,1]$ be given.

1.) $b^0_i := \mathbf{b}_i \ (i = 0, .., n)$
2.) Find the points b^1_i on $b^0_i \vee b^0_{i+1}$ from $t = cr(b^0_i, b^0_{i+1}, f_{i.i+1}, b^1_i)$! $(i = 0, .., n-1)$ (11)
3.) Find b^j_i from the above construction (10) ! $(j = 2, .., n;\ i = 0, .., n-j)$
4.) $x(t) := b^n_0$ (see fig.4)

(The degenerated case of several b^j_i being collinear has to be excluded in construction (10). It can be handled by mapping a non-degenerated situation to the degenerated one with help of a degenerated projective map.)
Both realizations are invariant under projective maps. Thus we have proved:

Theorem. *The relationship between rational curves and their Bézier polygon is projectively invariant.*

The control and weight points form a projectively invariant Bézier representation of a rational curve. This fact is the basis of the considerations in the next sections. Analogous assertions hold for dual Bézier curves (see [Hoschek'83]) and Bézier surfaces (cf. [Jüttler'92]). (In the case of Bézier surfaces, each mesh of the Bézier polygon net carries four interior (resp. exterior) weight points, but these weight points cannot be chosen arbitrarily: They have to be coplanar.)

2.3 Linear-fractional parameter transformations

A rational Bézier curve can be reparameterized by a linear-fractional parameter transformation

$$t^*(t) = \frac{t_0 \cdot t}{t_0 + (t-1)\cdot(2t_0 - 1)} \quad (t_0 \in \mathrm{I\!R} \cup \{\infty\}) \quad (12)$$

(cf. e.g. [Farin & Worsey'91], [Patterson'86]). This transformation preserves the control points $\underline{\mathbf{b}}_i$. It can be formulated in a geometric way: Let $w^*_{i.i+1}$ and $f^*_{i.i+1}$ be the points

Figure 5: A linear-fractional parameter transformation

Figure 6: The perspective collineation $\pi^{(i)}$: $\mathrm{p} \mapsto \mathrm{p}^*$

corresponding to a given parameter value t_0 and to $\frac{t_0}{2t_0-1}$ with respect to the projective scale (8) on $\mathbf{b}_i \vee \mathbf{b}_{i+1}$, respectively. The control points $\mathbf{b}_i$ $(i = 0, .., n)$ and the new weight points $\mathrm{w}^*_{i.i+1}$ and $\mathrm{f}^*_{i.i+1}$ $(i = 0, .., n-1)$ describe the reparameterized Bézier curve (fig. 5). For $t_0 = \infty$, the above transformation (12) permutes interior and exterior weight points and yields the complementary segment of the given curve.

3 Geometric continuity and perspective collineations

Now, a characterization of geometric continuity of second and third order by special projective maps is derived.
Certain perspective collineations preserve the curvature of a given curve at a point:

Lemma. *Let* $\mathrm{x} = \mathrm{x}(t)$ *be a curve,* $\mathrm{x}_0 = \mathrm{x}(t_0)$ *one of its points and* $\hat{\mathbf{t}} = \mathrm{x}(t_0) \vee \dot{\mathrm{x}}(t_0)$ *the tangent at* x_0. *The curve* x *is assumed to be at least four times continuously differentiable at* $t = t_0$. *Let* $\hat{\mathbf{a}}$ *be an arbitrary line* $(\neq \hat{\mathbf{t}})$ *through* x_0 *and* z *be an arbitrary point* $(\neq \mathrm{x}_0)$ *on* $\hat{\mathbf{t}}$, *and let* $\pi^{(i)}$, $\pi^{(ii)}$ *and* $\pi^{(iii)}$ *be perspective collineations*

- *(i) with centre* x_0 *and axis* $\hat{\mathbf{a}}$,
- *(ii) with centre* z *and axis* $\hat{\mathbf{t}}$ *and*
- *(iii) with centre* x_0 *and axis* $\hat{\mathbf{t}}$, *respectively.*

Then the curves $\mathrm{x}(t)$, $\pi^{(i)}(\mathrm{x}(t))$ *and* $\pi^{(ii)}(\mathrm{x}(t))$ *have a three-point-contact (i.e. a* G^2*-joint) and the curves* $\mathrm{x}(t)$ *and* $\pi^{(iii)}(\mathrm{x}(t))$ *even have a four-point-contact (i.e. a* G^3*-joint) at* x_0.

The figures 6, 7 and 8 show the perspective collineations $\pi^{(i)}$, $\pi^{(ii)}$ and $\pi^{(iii)}$ in the case of rational quadratic Bézier curves. Additionally, $t = 0$, i.e. $\mathrm{x}_0 = \mathbf{b}_0$ is chosen.

Proof. Instead of the occuring curves, their osculating conics at x_0 (i.e. the unique conics having a five-point-contact, cf. [Bol'50]) can be considered. (In [Geise & Nestler'91], Bézier representations of the conics having three-, four- or five-point-contacts with a given curve are derived.) Obviously, it is sufficient to prove the above assertion for conics x.
The set of the images $\pi^{(i)}(\mathrm{x})$ of the given conic x under all perspective collineations of type (i) forms a *pencil of conics*. All conics of such a pencil pass through four common points (which may be complex or multiple). Here, these four points are fixed points under the collineations $\pi^{(i)}$. Thus, they have to lie on the axis $\hat{\mathbf{a}}$. The conic x and the axis $\hat{\mathbf{a}}$ intersect in exactly two points: in x_0 and a second one p. The second intersection point p cannot be a multiple point of the pencil of conics as the perspective collineations $\pi^{(i)}$ do not preserve the tangent of x at p. Thus, x_0 is a triple point of the pencil of conics: All conics of the pencil have a three-point contact at x_0.

Figure 7: The perspective collineation $\pi^{(ii)}$: $\mathrm{p} \mapsto \mathrm{p}^*$

Figure 8: The perspective collineation $\pi^{(iii)}$: p↦ p*

The proof of case (ii) is dual to the above. Analogous deductions prove case (iii). ■

If in case (ii) the infinite point of the tangent $\hat{\mathbf{t}}$ is chosen as centre z, our lemma yields a well known result from the theory of polynomial Bézier curves: The control point $\mathbf{b}_2$ of a polynomial Bézier curve can be shifted along a parallel to $\mathbf{b}_0 \vee \mathbf{b}_1$ without influencing the curvature of the curve at $\mathbf{b}_0$.

In case of conics, the converse of the above lemma holds as well. Thus we have:

Theorem. *Two curves* $\mathrm{x}(t)$ *and* $\mathrm{y}(\tau)$ *have at a common point* $\mathrm{x}(t_0) = \mathrm{y}(\tau_0)$ *a* G^2*-joint (a* G^3*-joint), if and only if their osculating conics are connected by a perspective collineation of type (i) or (ii) (of type(iii)).*

Now, the perspective collineations $\pi^{(i)}$, $\pi^{(ii)}$, $\pi^{(iii)}$ will be used for the construction of geometrically continuous joints between rational Bézier curves.

4 Geometrically continuous joints between Bézier curves

With help of the perspective collineations of the above lemma, a construction of G^2-joints between Bézier curves is derived in this section. As an example, interpolating conic splines with second order continuity are discussed.
The third case of the above lemma allows to construct the *osculating parabola* of a conic (i.e. the unique parabola having a four-point-contact). Furthermore, necessary and sufficient conditions for the existence of an interpolating conic spline with third order continuity can be formulated.

4.1 Construction of G^2-joints

Let $\mathbf{b}_i$ $(i = -n,..,0)$ and $\mathrm{w}_{j.j+1}$ resp. $\mathrm{f}_{j.j+1}$ $(j = -n,..,-1)$ be the control points and the interior resp. exterior weight points of a given rational Bézier curve of degree n $(n \geq 2)$. This curve is to be continued by a second rational Bézier curve of degree n at its point $\mathbf{b}_0$. The control and weight points of the second curve are $\mathbf{b}_i$ $(i = 0,..,n)$ and $\mathrm{w}_{j.j+1}$ resp. $\mathrm{f}_{j.j+1}$ $(j = 0,..,n-1)$. The first control point $\mathbf{b}_1$ is assumed to lie on $\mathbf{b}_{-1} \vee \mathbf{b}_0$: This guarantees the first order continuity of both curves at $\mathbf{b}_0$. The weight points $\mathrm{w}_{0.1}$, $\mathrm{f}_{0.1}$ and $\mathrm{w}_{1.2}$, $\mathrm{f}_{1.2}$ are considered unknown. The control points $\mathbf{b}_{-2}$, $\mathbf{b}_{-1}$ and $\mathbf{b}_0$ resp. $\mathbf{b}_0$, $\mathbf{b}_1$ and $\mathbf{b}_2$ are assumed to be not collinear.
Both curves should join with second order continuity at $\mathbf{b}_0$. Meeting this requirement will produce the unknown weight points $\mathrm{w}_{0.1}$, $\mathrm{f}_{0.1}$ and $\mathrm{w}_{1.2}$, $\mathrm{f}_{1.2}$. They can be constructed as images of the given weight points $\mathrm{w}_{-2.-1}$, $\mathrm{f}_{-2.-1}$ and $\mathrm{w}_{-1.0}$, $\mathrm{f}_{-1.0}$ under two perspective collineations: The first perspective collineation $\pi^{(i)}$ has the centre $\mathbf{b}_0$ and the axis

$$\hat{\mathbf{a}} = \mathbf{b}_0 \vee ((\mathbf{b}_{-2} \vee \mathbf{b}_{-1}) \wedge (\mathbf{b}_1 \vee \mathbf{b}_2)) \tag{13}$$

Figure 9: The first perspective collineation

Figure 10: The second perspective collineation

and it maps $\mathbf{b}_{-1}$ into $\mathbf{b}_1$ (see fig.9). The points $\pi^{(i)}(\mathbf{b}_{-2})$, $\mathbf{b}_1 = \pi^{(i)}(\mathbf{b}_{-1})$ and $\mathbf{b}_2$ are collinear. The second perspective collineation $\pi^{(ii)}$ has the centre $\mathbf{b}_1$ and the axis $\mathbf{b}_{-1} \vee \mathbf{b}_0$ and it maps $\pi^{(i)}(\mathbf{b}_{-2})$ into $\mathbf{b}_2$ (see fig.10). The given curve and its image under $\pi^{(ii)} \circ \pi^{(i)}$ join with second order continuity at their common point $\mathbf{b}_0$ (see the above lemma!). The control and weight points of the image of the given curve are $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\pi^{(ii)}(\pi^{(i)}(\mathbf{b}_{-3}))$, ... and

$$\begin{aligned} \mathrm{w}_{0.1} &= \pi^{(ii)}(\pi^{(i)}(\mathrm{f}_{-1.0})), & \mathrm{f}_{0.1} &= \pi^{(ii)}(\pi^{(i)}(\mathrm{w}_{-1.0})), \\ \mathrm{w}_{1.2} &= \pi^{(ii)}(\pi^{(i)}(\mathrm{f}_{-2.-1})), & \mathrm{f}_{1.2} &= \pi^{(ii)}(\pi^{(i)}(\mathrm{w}_{-2.-1})), \ldots. \end{aligned} \tag{14}$$

(The perspective collineations $\pi^{(i)}$, $\pi^{(ii)}$ preserve the orientation of the given curve. Thus, the image of the complementary segment of the given curve is chosen as the new interior segment: Interior and exterior weight points are permuted.)
The control and weight points $\mathbf{b}_3$, $\mathbf{b}_4$, ... and $\mathrm{w}_{2.3}$, $\mathrm{f}_{2.3}$, $\mathrm{w}_{3.4}$, $\mathrm{f}_{3.4}$, ... do not influence the curvature of the second curve at $\mathbf{b}_0$. They can be chosen arbitrarily.
A possible realization of the whole construction is:

$$\begin{array}{lll}
\multicolumn{3}{l}{\text{First perspective collineation (see fig.9):}} \\
1.)\ \hat{\mathbf{a}} & := & \mathbf{b}_0 \vee ((\mathbf{b}_{-2} \vee \mathbf{b}_{-1}) \wedge (\mathbf{b}_1 \vee \mathbf{b}_2)) \\
2.)\ \mathrm{c} & := & (\mathbf{b}_{-2} \vee \mathbf{b}_0) \wedge (\mathbf{b}_1 \vee \mathbf{b}_2) \\
3.)\ \mathrm{h} & := & (\mathrm{f}_{-2.-1} \vee \mathbf{b}_0) \wedge (\mathbf{b}_1 \vee \mathbf{b}_2) \\
4.)\ \mathrm{w}_{0.1} & := & (\mathbf{b}_0 \vee \mathbf{b}_1) \wedge (\mathrm{h} \vee (\hat{\mathbf{a}} \wedge (\mathrm{f}_{-1.0} \vee \mathrm{f}_{-2.-1}))) \\
\multicolumn{3}{l}{\text{Second perspective collineation (see fig.10):}} \\
\multicolumn{3}{l}{\text{Let } \hat{\mathbf{g}} \text{ be an arbitrary line} \neq \mathbf{b}_0 \vee \mathbf{b}_1 \text{ through } \mathbf{b}_1 \text{ !}} \\
5.)\ \mathrm{p} & := & (\mathbf{b}_0 \vee \mathrm{c}) \wedge \hat{\mathbf{g}} \\
6.)\ \mathrm{q} & := & (\mathbf{b}_0 \vee \mathbf{b}_2) \wedge \hat{\mathbf{g}} \\
7.)\ \mathrm{r} & := & (\mathrm{p} \vee \mathrm{h}) \wedge (\mathbf{b}_0 \vee \mathbf{b}_1) \\
8.)\ \mathrm{w}_{1.2} & := & (\mathrm{q} \vee \mathrm{r}) \wedge (\mathbf{b}_1 \vee \mathbf{b}_2)
\end{array} \tag{15}$$

(A permutation of f with w yields the construction of the unknown exterior weight points. The figures 9 and 10 shows the construction (15) in the case of rational quadratic curves, i.e. for $n = 2$.)
The solution of the problem (to find the unknown weight points $\mathrm{w}_{0.1}$, $\mathrm{f}_{0.1}$ and $\mathrm{w}_{1.2}$, $\mathrm{f}_{1.2}$) is uniquely determined up to linear-fractional parameter transformations (12). This results e.g. from Boehm's formula for the curvature κ of a rational Bézier curve at its first control point (see [Boehm'87]) or from the continuity constraints developed in [Degen'88]. Thus we have:

Theorem. *The two given curves described by the control and weight points* $\mathbf{b}_{-n}$, ..., $\mathbf{b}_0$*;* $\mathrm{w}_{-n.-n+1}$, $\mathrm{f}_{-n.-n+1}$, ..., $\mathrm{w}_{-1.0}$, $\mathrm{f}_{-1.0}$ *and* $\mathbf{b}_0$, ..., $\mathbf{b}_n$*;* $\mathrm{w}_{0.1}$, $\mathrm{f}_{0.1}$, ..., $\mathrm{w}_{n-1.n}$, $\mathrm{f}_{n-1.n}$ *(where* $\mathbf{b}_0$, $\mathbf{b}_{-1}$ *and* $\mathbf{b}_1$ *are assumed to be collinear and* $\mathbf{b}_{-2}$, $\mathbf{b}_{-1}$ *and* $\mathbf{b}_0$ *to be not collinear) have at* $\mathbf{b}_0$ *a three-point-contact (i.e. a* G^2*-joint) if and only if the weight points* $\mathrm{w}_{0.1}$, $\mathrm{f}_{0.1}$ *and* $\mathrm{w}_{1.2}$, $\mathrm{f}_{1.2}$ *result from the weight points* $\mathrm{w}_{-1.0}$, $\mathrm{f}_{-1.0}$ *and* $\mathrm{w}_{-2.-1}$, $\mathrm{f}_{-2.-1}$ *by construction (15) and a linear-fractional parameter transformation (12).*

In figure 11a, an interpolating G^2-conic spline is constructed using the above construction

a) One step of construction of the spline
b) $w_1 = 0.5$
c) $w_1 = 1.0$
d) $w_1 = 2.0$

Figure 11: The construction of an interpolating G^2-conic spline

Figure 12: The construction of the osculating parabola of a conic

(15). The conic segments are described by rational Bézier curves of degree two. The spline interpolates the four tangent elements " $\mathbf{b}_0 \vee \mathbf{b}_1$ *plus* $\mathbf{b}_0$ ", " $\mathbf{b}_1 \vee \mathbf{b}_3$ *plus* $\mathbf{b}_2$ ", " $\mathbf{b}_3 \vee \mathbf{b}_5$ *plus* $\mathbf{b}_4$ " and " $\mathbf{b}_5 \vee \mathbf{b}_6$ *plus* $\mathbf{b}_6$ ". In fig.11, the first segment of the conic spline is given in standard form ($w_0 = w_2 = 1$). The weight w_1 of the first control point can be chosen arbitrarily. The figures 11b,c,d illustrate the influence of w_1: By increasing w_1, the whole spline curve is pulled to the edges of the control polygon. The weight w_1 can be viewed as a *global tension parameter* of the curve.

4.2 The osculating parabola of a conic

Polynomial Bézier curves of degree two are parabolas (i.e. the infinite line is a tangent of these curves). Let a rational curve of degree two with the control and weight points $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathrm{w}_{0.1}$, $\mathrm{f}_{0.1}$, $\mathrm{w}_{1.2}$, $\mathrm{f}_{1.2}$ be given. The three control points $\mathbf{b}_0$, $\mathbf{b}_1$ and $\mathbf{b}_2$ are assumed to be not collinear.
The given curve is a parabola iff $\frac{w_2}{w_1} = \frac{w_1}{w_0}$ (cf. e.g. [Lee'87]). This condition can be expressed in terms of ratios of the control and weight points:

$$\frac{dist(\mathbf{b}_2, \mathrm{w}_{1.2})}{dist(\mathrm{w}_{1.2}, \mathbf{b}_1)} = \frac{dist(\mathbf{b}_1, \mathrm{w}_{0.1})}{dist(\mathrm{w}_{0.1}, \mathbf{b}_0)} \quad . \tag{16}$$

Generally, this condition is not fulfilled.
Consider perspective collineations with centre $\mathbf{b}_0$ and axis $\mathbf{b}_0 \vee \mathbf{b}_1$. There exists a unique perspective collineation $\pi^{(iii)}$ mapping the given conic to a parabola (see fig.12):

$$\begin{array}{ll} 1.) & \text{Find the point p on } \mathbf{b}_0 \vee \mathbf{b}_1 \text{ from } dist(\mathbf{b}_0, \mathrm{p}) = dist(\mathrm{w}_{0.1}, \mathbf{b}_1)\ ! \\ 2.) & \text{Construct the parallel } \hat{\mathrm{l}} \text{ to } \mathbf{b}_0 \vee \mathbf{b}_2 \text{ through p !} \\ 3.) & \mathrm{w}^*_{1.2} := \hat{\mathrm{l}} \wedge (\mathbf{b}_0 \vee \mathrm{w}_{1.2}) \\ 4.) & \mathrm{b}^*_2 := (\mathbf{b}_1 \vee \mathrm{w}^*_{1.2}) \wedge (\mathbf{b}_0 \vee \mathbf{b}_2) \\ 5.) & \mathrm{b}^*_0 := \mathbf{b}_0, \quad \mathrm{w}^*_{0.1} := \mathrm{w}_{0.1}, \quad \mathrm{b}^*_1 := \mathbf{b}_1 \end{array} \tag{17}$$

The image of the given curve under the perspective collineation $\pi^{(iii)}$ has the control and weight points b^*_0, b^*_1, b^*_2 and $\mathrm{w}^*_{0.1}$, $\mathrm{f}^*_{0.1}$, $\mathrm{w}^*_{1.2}$, $\mathrm{f}^*_{1.2}$. Resulting from 1.) and from the intercept theorems, it fulfills condition (16): The image curve is a parabola. The given curve and its image under $\pi^{(iii)}$ have a four-point-contact at $\mathbf{b}_0$ (see the above lemma). Thus, the image curve is the *osculating parabola* of the given conic at $\mathbf{b}_0$.
It can be reparametrized by linear-fractional parameter transformations (12). One of the

Figure 13: The relation between $\mathbf{b}_2$ and the number of poles

possible parametrizations is a polynomial one: The interior weight points are the midpoints of the segments and the exterior weight points are the infinite points of the lines of the Bézier polygon.

4.3 The existence of an interpolating G^3-conic spline

Let a conic segment with the control and weight points $\mathbf{b}_{-2}$, $\mathbf{b}_{-1}$, $\mathbf{b}_0$ and $\mathrm{w}_{-2.-1}$, $\mathrm{f}_{-2.-1}$, $\mathrm{w}_{-1.0}$, $\mathrm{f}_{-1.0}$ be given. This curve is to be continued by a second conic segment at its point $\mathbf{b}_0$. One point of this second conic segment is assumed to be known, it is chosen as control point $\mathbf{b}_2$. Both curves should have a four-point-contact (i.e. a G^3-joint) at their common point $\mathbf{b}_0$. The second conic segment is uniquely determined by these constraints.
Generally, the second segment may intersect the infinite line. Such intersection points (i.e. poles) are undesired in practical applications. How does the number of poles depend on the given point $\mathbf{b}_2$?
The answer to this question is given in fig.13: If $\mathbf{b}_2 \in M_i$, then the second conic segment has exactly i poles ($i = 0, 1, 2$). The auxiliary curve p is the osculating parabola of the given conic segment at $\mathbf{b}_0$.
This answer follows immediately by considering the perspective collineations with axis $\mathbf{b}_{-1} \vee \mathbf{b}_0$ and centre $\mathbf{b}_0$. They generate all possible second segments. The osculating parabola p separates the solutions of our interpolation problem with respect to the number of its poles.

Conclusion

The constructions concerning geometric continuity of second order can be directly generalized to non-planar curves: They then take place in the osculating plane of the occuring curves. In [Jüttler'92], geometric realizations of the algorithms for subdivision and degree elevation of rational Bézier curves are derived. Furthermore, the de Casteljau-algorithm is shown to be a generalization of Steiner's generation of conics. In this sense, rational Bézier curves of arbitrary degree are *generalized conics.*
The authors believe that the constructive-geometric considerations of this paper allow to develop a better qualitative understanding of the rational Bézier technique. Obviously these considerations cannot (and they do not intend to) replace the analytical discussion of rational Bézier curves. But the power of the pure constructive-geometrical methods is quite surprising. For example, the analytical discussion of the conditions for the existence of an interpolating third order continuous conic spline seems to be more expenditious than the geometrical approach presented in this paper.

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