# A Geometrical Approach to Interpolation on Quadric Surfaces

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**Abstract.** The paper presents a very powerful construction for rational curves and surfaces on quadrics. Based on a result of number theory, a generalization of the stereographic projection is introduced in the case of the unit sphere. With help of this map, interpolation with spherical rational curves is shown to be a linear problem. The existence of a quadratic triangular and of a biquadratic tensor–product Bézier patch on the sphere interpolating given boundaries is discussed. The final section outlines the extension to arbitrary non–degenerated quadric surfaces.

## §1. Introduction

Quadric surfaces (like ellipsoids, hyperboloids of one or two sheets, etc.) play traditionally an important role in several industrial applications. In order to include them into computer-aided design systems, a mathematical description of curve segments and of surface patches on quadrics is required. Rational parametric representations of curves and surfaces ( *e.g.*, NURBS-curves and -surfaces) support the exact description of conic sections and quadric surfaces.

In recent years, several authors have developed different constructions for rational curves and surfaces on quadrics [1,8,9,12,14,...]. Most of these papers are based on the use of the classical stereographic projection.

In the case of the unit sphere $U$, the stereographic projection $\sigma$ connects the points of the equator plane $P$ $(z = 0)$ with those of the sphere. The north pole $\mathbf{z}$ of the sphere is chosen as the centre of projection. The line connecting the north pole with an arbitrary point $\mathbf{p}$ of the equator plane intersects the sphere in $\mathbf{z}$ and in a second intersection point $\sigma(\mathbf{p})$ (see Fig. 1).

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The map $\mathbf{p} \mapsto \sigma(\mathbf{p})$ from the equator plane $P$ to the unit sphere $U$ is called the *stereographic projection* with centre $\mathbf{z}$. It will be described with help of homogeneous coordinates. These coordinates are defined by the relation

$$1 : x : y : z = x_0 : x_1 : x_2 : x_3 \tag{1}$$

(see [2]), and the bold letter $\mathbf{x} = (\, x_0\ x_1\ x_2\ x_3\,)^\top$ denotes the homogeneous coordinates of a point in three-dimensional space.

The image of a point $\mathbf{p} = (\, p_0\ p_1\ p_2\ 0\,)^\top$ on the equator plane $P$ under the stereographic projection $\sigma$ is the point

$$\sigma(\mathbf{p}) = \begin{pmatrix} p_0^2 + p_1^2 + p_2^2 \\ 2p_0p_1 \\ 2p_0p_2 \\ p_1^2 + p_2^2 - p_0^2 \end{pmatrix} \tag{2}$$

on the sphere. The image of a rational curve (resp. surface) on the equator plane under $\sigma$ is a rational curve (resp. surface) on the sphere. But the stereographic projection has an important disadvantage: *it does not yield all irreducible rational curves and surfaces on quadrics.* (A rational curve resp. surface is said to be irreducible, if its homogeneous coordinates do not have any common linear factors, *i.e.*, if $\gcd(x_0, x_1, x_2, x_3) = 1$ holds.) For instance, it is impossible to construct all biquadratic rational Bézier surface patches as the images of bilinear patches under a stereographic projection. (Some counterexamples have been found by Geise and Langbecker [9], by Boehm and Hansford [1], and by Fink [8].)

The present paper deals with a generalization of the stereographic projection. Based on a result of number theory, the generalized stereographic projection has been developed by Dietz et al. [5,6]. This paper discusses some geometrical aspects of the method.

At first, the generalized stereographic projection is introduced in the case of the unit sphere as a representative of the class of oval quadric surfaces (ellipsoids, hyperboloids of two sheets, etc.). Then, the method is applied to interpolation with spherical rational curves and to the construction of rational surface patches on the sphere. The final section outlines the extension of the results to the hyperbolic paraboloid as a representative of the class of doubly–ruled quadric surfaces (hyperbolic paraboloids and hyperboloids of one sheet).

## §2. The Generalized Stereographic Projection

Using homogeneous coordinates (1), the unit sphere $U$ is given by the implicit equation

$$u_0^2 = u_1^2 + u_2^2 + u_3^2. \tag{3}$$

The following considerations are based on an algebraic approach to rational curves and surfaces on the sphere as introduced in [11]. The homogeneous coordinates of a rational curve resp. of a rational tensor–product surface on the sphere have to satisfy equation (3). Thus, this curve resp. this surface can be considered as a solution of the diophantic equation (3) in the ring of polynomials $\mathbb{R}[t]$ resp. in the ring of bivariate polynomials $\mathbb{R}[u, v]$.

Already in 1868, V. A. Lebesgue has found the representation formula

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} r_0^2 + r_1^2 + r_2^2 + r_3^2 \\ 2r_0r_1 - 2r_2r_3 \\ 2r_1r_3 + 2r_0r_2 \\ r_1^2 + r_2^2 - r_0^2 - r_3^2 \end{pmatrix} =: \delta(\mathbf{r}) \tag{4}$$

for *all irreducible solutions* of equation (3) in the ring of integers [3]. This formula can be directly generalized to arbitrary polynomial rings [5].

Of course, a geometric interpretation of the representation formula (4) would be very helpful for the construction of rational curves and surfaces on quadrics. So we define a map:

**Definition 1.** *The map $\mathbf{r} \in \bar{E}^3 \mapsto \delta(\mathbf{r}) \in U$ from the three-dimensional space (which is projectively completed by adding points at infinity) to the unit sphere $U$ (see (4)) is called the generalized stereographic projection.*

Now, the algebraic properties of the representation formula (4) can be formulated geometrically: *any irreducible spherical rational curve of polynomial degree $2n$ (resp. spherical rational tensor–product surface of degree $(2m, 2n)$) can be constructed as the image of a spatial rational curve of polynomial degree $n$ (resp. of a spatial rational tensor–product surface of degree $(m, n)$) under the generalized stereographic projection $\delta$.*

For $r_3 = 0$, the representation formula (4) yields exactly the classical stereographic projection (2). Thus we have the following

**Proposition 2.** *The restriction of the generalized stereographic projection $\delta : \bar{E}^3 \to U$ to the equator plane $r_3 = 0$ of the unit sphere $U$ is the stereographic projection $\sigma : P \to U$.*

Of course, the properties of the classical stereographic projection $\sigma$ are well known. For example, this map preserves circles, *i.e.*, the image of a circle or a line on the equator plane under the stereographic projection is a circle on the sphere. In order to discuss the generalized stereographic projection, this map is decomposed into the classical stereographic projection and an auxiliary map $\theta$.

**Theorem 3.** *The generalized stereographic projection $\delta : \bar{E}^3 \to U$ is the composition of the hyperbolic projection $\theta : \mathbf{r} \in \bar{E}^3 \mapsto \theta(\mathbf{r}) \in P$, where*

$$\theta(\mathbf{r}) = \begin{pmatrix} r_0^2 + r_3^2 \\ r_0r_1 - r_2r_3 \\ r_1r_3 + r_0r_2 \\ 0 \end{pmatrix}, \tag{5}$$

*with the stereographic projection $\sigma : P \to U$, i.e., $\delta(\mathbf{r}) = \sigma(\theta(\mathbf{r}))$.*

The proof results from straightforward calculations. Figure 1 shows the generalized stereographic projection and its decomposition. At first, the points of the three-dimensional space are mapped to the equator plane $P$ by the hyperbolic projection, and then they are mapped to the unit sphere $U$ with help of the stereographic projection.

**Figure 1.** The generalized stereographic projection.

The discussion of the hyperbolic projection starts with the inverse image of a point.

**Proposition 4.** *The inverse image of a point* $\mathbf{p} = (\, p_0 \; p_1 \; p_2 \; 0 \,)^\top$ *of the equator plane* $P$ *under the hyperbolic projection* $\theta$ *(see (5)) is the line*

$$\lambda \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ p_2 \\ -p_1 \\ p_0 \end{pmatrix} \quad (\lambda, \mu \in \mathbb{R}) \tag{6}$$

*in three-dimensional space. This line intersects the equator plane in its image under* $\theta$*, so it will be called a* projecting line *of the hyperbolic projection. Resulting from Theorem 3, the inverse image of a point* $\mathbf{u} = (\, u_0 \; u_1 \; u_2 \; u_3 \,)^\top$ *under the generalized stereographic projection* $\delta$ *is the line (6), where* $\mathbf{p} = \sigma^{-1}(\mathbf{u}) = (\; (u_0 - u_3) \;\; u_1 \;\; u_2 \;\; 0 \,)^\top$.

The projecting lines (6) of the hyperbolic projection are located on hyperboloids of revolution around the $z$–axis. They form an *elliptic linear congruence of lines* [10] (or a *net* of lines), *i.e.*, they pass through two distinct focal lines, which are both conjugate–complex and at infinity. *The hyperbolic projection* $\theta$ *(see (5)) is a special net projection*, *i.e.*, a projection with respect to a net of lines.

The net projection has been introduced by Tuschel in 1911 in order to develop a constructive geometry for helices [13]. A second approach to the net projection has been discovered by Wunderlich in 1936: it can be considered as a non–Euclidean parallel projection [15].

The next proposition summarizes some properties of the hyperbolic projection (cf. [7]) and the resulting properties of the generalized stereographic projection.

**Proposition 5.** *The image of an arbitrary non–projecting line under the hyperbolic projection is a circle or a line on the equator plane $P$. Its image under the generalized stereographic projection is a circle on the sphere (because the stereographic projection preserves circles).*

*The inverse image of a circle on the sphere under the generalized stereographic projection is a ruled surface formed by projecting lines. This ruled surface proves out to be either a hyperbolic paraboloid or a hyperboloid of one sheet.*

*Any plane in three-dimensional space contains exactly one projecting line (6) of the hyperbolic projection. Thus, the generalized stereographic projection maps the lines of a fixed plane to the circles through one fixed point. This point corresponds to the projecting line contained in the given plane.*

The special role of the circles in the equator plane $P$ results from the fact, that the two focal lines of the elliptic linear congruence of lines intersect the equator plane in the two circular points at infinity.

The next section applies the generalized stereographic projection to the construction of an interpolating spherical rational curve.

## §3. Interpolation with Spherical Rational Curves

Let $m+1$ points $\mathbf{p}_i$ on the unit sphere $U$ with parameters $t_i \in \mathbb{R}$ $(i = 0, .., m)$ be given. These points are to be interpolated by a spherical rational curve $\mathbf{x}(t)$. Such a curve can be constructed with help of the following algorithm:

1. Find the inverse images of the given points $\mathbf{p}_i$ under the generalized stereographic projection (cf. Proposition 4)! These inverse images are certain projecting lines in three-dimensional space.
2. Construct a spatial rational curve $\mathbf{y}(t)$ which passes through the inverse images of the given points! The point $\mathbf{y}(t_i)$ has to be located on the inverse image of the given point $\mathbf{p}_i$. The spatial curve $\mathbf{y}(t)$ can be found by solving a linear system of equations.
3. Apply the generalized stereographic projection $\delta$ to the spatial curve $\mathbf{y}(t)$! Its image is the required interpolating curve $\mathbf{x}(t)$ on the unit sphere.

Note that the interpolating spherical curve is found by solving a linear system of equations. Thus, interpolation with rational curves on quadrics proves out to be a *linear* problem. The details of the method and some properties of the obtained solution will be discussed in [4].

## §4. Rational Surface Patches on the Sphere

This section briefly discusses the construction of quadratic triangular and of biquadratic tensor–product surface patches on the sphere. Any spherical quadratic triangular patch

$$\mathbf{x}(u,v,w) = \sum_{i+j+k=2} B^2_{i,j,k}(u,v,w)\,\mathbf{c}_{i,j,k} \quad (u,v,w \geq 0;\ u+v+w=1) \qquad (7)$$

(where the $B^n_{i,j,k}(u,v,w) = \frac{n!}{i!j!k!}u^i v^j w^k$ denote the trivariate Bernstein polynomials and the $\mathbf{c}_{i,j,k} \in \mathbb{R}^4$ are the homogeneous control points) can be constructed as the image of a *linear* triangular patch under the generalized stereographic projection. The boundaries of this linear patch are three lines in three-dimensional space, and these three lines are contained in a plane. Resulting from the third part of Proposition 5, the three boundary circles of the quadratic triangular patch (7) (which are the images of the three boundary lines of the linear patch under $\delta$) must intersect in one point.

**Theorem 6.** *If the quadratic triangular surface patch (7) is part of the sphere, then its three boundaries intersect in one point. Conversely, if three spherical circles satisfy this condition, then they can be interpolated by a spherical quadratic triangular patch.*

This theorem has been discovered by Sederberg and Anderson by discussing Steiner surface patches on quadrics [12].

Now, the method is applied to biquadratic tensor–product surface patches on the sphere. Any spherical biquadratic patch

$$\mathbf{y}(u,v) = \sum_{i=0}^{2}\sum_{j=0}^{2} B^2_i(u)B^2_j(v)\,\mathbf{b}_{i,j} \quad ((u,v) \in [0,1]\times[0,1]) \qquad (8)$$

(where the $B^n_i(t) = \binom{n}{i}t^i(1-t)^{n-i}$ denote the Bernstein polynomials and the $\mathbf{b}_{i,j} \in \mathbb{R}^4$ are the homogeneous control points) can be constructed as the image of a *bilinear* patch under the generalized stereographic projection. The four boundary circles of the spherical patch (8) intersect in the four corner points $\mathbf{p}_i$ and in four second intersection points $\mathbf{q}_i$ (i=1,..,4), see Fig. 2.

In [5], a condition analogous to that of Theorem 6 has been derived:

**Theorem 7.** *If the biquadratic patch (8) is part of the sphere, then the four points $\mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_3, \mathbf{q}_4$ (or equivalently $\mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_3, \mathbf{p}_4$) are located on one circle (see Fig. 2). Conversely, if four spherical circles satisfy this condition, then they can be interpolated by a spherical biquadratic patch.*

Using the generalized stereographic projection, this theorem can be proved directly (see [5]). The four control points of the bilinear patch (which is the preimage of the biquadratic patch) span two planes, and the circle from Theorem 7 is the image of the line, in which the two planes intersect.

The equivalence of the existence of the two circles in Theorem 7 is known as *Miquel's Theorem* in the foundations of geometry.

**Figure 2.** The boundaries of the spherical biquadratic patch.
(a) Scheme of the boundaries, (b) The spherical patch.

## §5. Extension to Other Quadric Surfaces

This section outlines the discussion of the hyperbolic paraboloid

$$h_0 h_3 = h_1 h_2 \tag{9}$$

as a representative of doubly–ruled quadric surfaces. In [5], the representation formula

$$\begin{aligned} h_0 &= r_0 r_3 \qquad & h_1 &= r_1 r_3 \\ h_2 &= r_0 r_2 \qquad & h_3 &= r_1 r_2 \end{aligned} \tag{10}$$

has been given which is analogous to that of the unit sphere (2).

Again, this formula is considered as a generalized stereographic projection. This map proves out to be the composition of a net projection (with respect to a hyperbolic linear congruence of lines) with a stereographic projection. Rational curves and surfaces on the hyperbolic paraboloid can be constructed similarly to the case of the unit sphere, see [6].

Any non–degenerated oval resp. doubly–ruled quadric surface is the image of the unit sphere resp. of the hyperbolic paraboloid under an appropriate projective map. (For instance, this map can be constructed with help of a principal axes transformation, see [6].) Thus, the methods and results of this paper can be directly generalized to arbitrary non–degenerated quadric surfaces.

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