Short communication

A vegetarian approach to optimal parameterizations

Bert Jüttler

Fachbereich Mathematik, Technische Universität Darmstadt,
Schloßgartenstraße 7, 64289 Darmstadt, Germany
juettler@mathematik.th-darmstadt.de

Abstract. According to a recent result by Farouki (1997), the optimal bilinear parameter transformation of an integral Bézier curve (which produces a rational parameterization whose parametric speed is as uniform as possible) can be computed by solving a quadratic equation. This note presents a simplified derivation of this result. In addition we outline its generalization to rational curves.

Keywords. Rational parameterization, arc length parameterization.

The article (Farouki, 1997) studies bilinear parameter transformations of integral Bézier curves. After introducing an optimality criterion which measures the deviation from the uniform speed parameterization, it is discussed how the optimal rational representation (which is shown to be unique) of an integral Bézier curve can be found. The computation of the optimal representation is reduced to finding the root of a quadratic equation. The derivation of these results, however, involves some “meaty” calculations, and Footnote 5 even advises vegetarians to skip from some of the equations.

In this note we present a simplified approach to the above results. As remarked on page 161 of (Farouki, 1997), the existence of such an approach is suggested by the elementary final form of the equation for the optimal parameterization variable.

Consider an integral Bézier curve

$$\mathbf{x}(t) = \sum_{i=0}^{n} B_i^n(t)\, \mathbf{p}_i, \quad t \in [0,1], \tag{1}$$

of degree n with the control points $\mathbf{p}_i \in \mathbb{R}^2$ and the well–known Bernstein polynomials $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$, see (Hoschek and Lasser, 1993). We choose a fixed parameter α,

$0 < \alpha < 1$, and apply the bilinear parameter transformation

$$u \mapsto t_\alpha(u) = \frac{u\,(\alpha - 1)}{2\,u\,\alpha\; - u\; - \alpha} \tag{2}$$

to the curve (1). The transformation maps the unit interval onto itself, in particular it satisfies $t_\alpha(0) = 0$, $t_\alpha(1) = 1$ and $t_\alpha(\frac{1}{2}) = 1-\alpha$. The inverse transformation of (2) evaluates to

$$t \mapsto u_\alpha(t) = \frac{\alpha\, t}{1 - \alpha + 2\,\alpha\, t - t}. \tag{3}$$

Applying the transformation $t_\alpha(u)$ to the curve (1) results in the rational Bézier curve

$$\mathbf{x}(t_\alpha(u)) = \frac{\sum\limits_{i=0}^{n} B_i^n(u)\; w_i\; \mathbf{p}_i}{\sum\limits_{i=0}^{n} B_i^n(u)\; w_i} \tag{4}$$

with the new curve parameter $u \in [0, 1]$ and with the weights $w_i = (1-\alpha)^i\; \alpha^{n-i}$. Let S denote the total arc length of the curve,

$$S = \int_0^1 \|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\|\mathrm{d}t = \int_0^1 \|\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{x}(t_\alpha(u))\|\mathrm{d}u. \tag{5}$$

Of course, the bilinear parameter transformation preserves the arc length of the curve. As proposed by (Farouki, 1997), we want to choose the re–parameterization (2) (which is governed by the parameter α) such that the value of

$$\int_0^1 \left(\|\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{x}(t_\alpha(u))\| - S\right)^2 \mathrm{d}u \tag{6}$$

becomes as small as possible. This expression measures the deviation of $\mathbf{x}(t_\alpha(u))$ from the uniform–speed parameterization of the curve. As an immediate consequence from (5) and (6), this is equivalent to finding the minimum of the function

$$J(\alpha) = \int_0^1 \|\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{x}(t_\alpha(u))\|^2\,\mathrm{d}u. \tag{7}$$

With the help of the chain rule we get from (7)

$$J(\alpha) = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}u}t_\alpha(u)\right)^2 \|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t_\alpha(u))\|^2\,\mathrm{d}u. \tag{8}$$

By substituting $u = u_\alpha(t)$ we transform the integration variable back to the original curve parameter t,

$$J(\alpha) = \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}u}t_\alpha(u)\right|_{u=u_\alpha(t)} \|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\|^2\quad \mathrm{d}t. \tag{9}$$

The derivative of the parameter transformation $t_\alpha(u)$ can be expressed with the help of the inverse parameter transformation $u_\alpha(t)$,

$$J(\alpha) = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t} u_\alpha(t)\right)^{-1} \|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\|^2 \,\mathrm{d}t. \tag{10}$$

Now Eq. (3) gives the final result for $J(\alpha)$,

$$J(\alpha) = \int_0^1 \frac{(1-\alpha+2\,t\,\alpha-t)^2}{\alpha\,(1-\alpha)} \;\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\|^2 \,\mathrm{d}t. \tag{11}$$

We assume that the curve (1) is regular, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) \neq \mathbf{0}$ holds for $t \in [0,1]$. Thus, there exist two constants C_1, C_2 with

$$0 < C_1 < \|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\| < C_2 \quad \text{for} \quad t \in [0,1]. \tag{12}$$

Resulting from

$$\int_0^1 \frac{(1-\alpha+2\,t\,\alpha-t)^2}{\alpha\,(1-\alpha)}\,\mathrm{d}t = \frac{1}{3}\,\frac{\alpha^2-\alpha+1}{\alpha\,(1-\alpha)} \tag{13}$$

we obtain from (11) the inequalities

$$C_1\,\frac{1}{3}\,\frac{\alpha^2-\alpha+1}{\alpha\,(1-\alpha)} < J(\alpha) < C_2\,\frac{1}{3}\,\frac{\alpha^2-\alpha+1}{\alpha\,(1-\alpha)}. \tag{14}$$

Note that

$$\lim_{\alpha\to 0,\alpha>0} \frac{\alpha^2-\alpha+1}{\alpha\,(1-\alpha)} = \lim_{\alpha\to 1,\alpha<1} \frac{\alpha^2-\alpha+1}{\alpha\,(1-\alpha)} = +\infty. \tag{15}$$

Hence the problem $J(\alpha) \to \mathrm{Min}$ possesses a global minimum $\alpha = \alpha_0$ for $0 < \alpha < 1$. Moreover, this solution satisfies the equation

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}\alpha}\,J(\alpha) \;=\; \int_0^1 \frac{(2\,t-1)\,\alpha^2 + 2\,(1-t)^2\,\alpha - (1-t)^2}{\alpha^2\,(1-\alpha)^2}\,\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\|^2 \,\mathrm{d}t, \tag{16}$$

cf. (11). We express the numerator of the fraction in Bernstein–Bézier form,

$$\begin{aligned} 0 \;=\; \frac{\mathrm{d}}{\mathrm{d}\alpha}\,J(\alpha) \;&=\; \int_0^1 \frac{-B_0^2(t)\;B_0^2(\alpha) + B_2^2(t)\;B_2^2(\alpha)}{\alpha^2\,(1-\alpha)^2}\,\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\|^2 \,\mathrm{d}t \\ &=\; \frac{P\,B_0^2(\alpha) + Q\,B_2^2(\alpha)}{\alpha^2\,(1-\alpha)^2}, \end{aligned} \tag{17}$$

with the coefficients

$$P = \int_0^1 -B_0^2(t)\;\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\|^2\,\mathrm{d}t \quad \text{and} \quad Q = \int_0^1 B_2^2(t)\;\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\|^2\,\mathrm{d}t. \tag{18}$$

Note that $P < 0 < Q$ holds, as the Bernstein polynomials are non–negative on the unit interval. Hence, we get exactly one root of $P\,B_0^2(\alpha) + Q\,B_2^2(\alpha) = 0$ with $0 < \alpha < 1$.

Using the identities

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{x}(t) = n \sum_{i=0}^{n-1} B_i^{n-1}(t)\,\Delta\mathbf{p}_i \tag{19}$$

with $\Delta\mathbf{p}_i = \mathbf{p}_{i+1} - \mathbf{p}_i$,

$$\int_0^1 B_i^n(t)\,\mathrm{d}t = \frac{1}{n+1}, \tag{20}$$

and the product formulas for Bernstein polynomials (see (Farouki and Rajan, 1988) or (Hoschek and Lasser, 1993)) we compute the coefficients P and Q,

$$\begin{aligned} P &= -\frac{n^2}{2n+1} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\binom{n-1}{i}\binom{n-1}{j}}{\binom{2n}{i+j}}\,\Delta\mathbf{p}_i \cdot \Delta\mathbf{p}_j \quad \text{and} \\ Q &= \frac{n^2}{2n+1} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\binom{n-1}{i}\binom{n-1}{j}}{\binom{2n}{i+j+2}}\,\Delta\mathbf{p}_i \cdot \Delta\mathbf{p}_j. \end{aligned} \tag{21}$$

The parameter $\alpha = \alpha_0$ of the optimal parameter transformation (2) can now easily be obtained as the unique root of the quadratic polynomial

$$0 = P\,B_0^2(\alpha) + Q\,B_2^2(\alpha) = P\,(1-\alpha)^2 + Q\,\alpha^2 \tag{22}$$

with $0 < \alpha < 1$.

Note that this approach can be generalized to the case of rational Bézier curves, cf. Section 5 of (Farouki, 1997). The calculations from Eq. (6) till Eq. (18) are valid for rational Bézier curves too. Hence, the parameter α of the optimal re–parameterization (2) of a rational Bézier curve is again the unique root of (22) with $0 < \alpha < 1$. Unlike the polynomial case, the coefficients P and Q have to be found by integrating the rational (rather than polynomial) expressions (18). In practice one should use numerical quadratures to evaluate these coefficients, as the exact integration via partial–fraction decomposition seems to be too expensive.

References

Farouki, R. T., and Rajan, V. T. (1988), Algorithms for polynomials in Bernstein form, Computer Aided Geometric Design 5, 1–26.

Farouki, R. T. (1997), Optimal parameterizations, Computer Aided Geometric Design 14, 153–168.

Hoschek, J., and Lasser, D. (1993), *Fundamentals of Computer Aided Geometric Design*, AK Peters, Wellesley.