ARBITRARILY WEAK LINEAR CONVEXITY CONDITIONS FOR MULTIVARIATE POLYNOMIALS

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Abstract

We present a general construction of linear sufficient convexity conditions for multivariate polynomials in Bernstein–Bézier representation over simplices. As the main new feature of the construction, the obtained conditions can be made as weak as desired; they can be adapted to any finite set of strongly convex polynomials.

1. Introduction

Convexity criteria for multivariate polynomials in Bernstein–Bézier (BB) representation over simplices have attracted a great deal of research during the last years, see e.g. the surveys by Goodman [8] and Dahmen [4]. For bivariate polynomials it has been observed by Chang and Davis [2] at first, that convexity of the control net (or, more precisely, convexity of the piecewise linear interpolant to the Bézier coefficients) of a polynomial implies convexity of the polynomial. This result was generalized to the multivariate case by Dahmen and Micchelli [5]. Note that in the general multivariate case there is no unique piecewise linear interpolant of the Bézier coefficients. This is due to the fact that for dimension ≥ 3 no canonical decomposition of a simplex into smaller simplices exists, see e.g. Figure 3 in [5].

Of course, convexity of the control net is far from being a necessary condition for convexity of the bivariate polynomial. In order to find weaker criteria, several authors investigated the effects of (artificial) degree raising [5, 14] and subdivision [6, 9, 10, 11] of the BB representation. As shown by Gregory and Zhou [11], subdivision preserves the convexity of the control net if and only if one uses regular partitions of the original domain triangle (the edges of the sub-triangles have to be parallel to the ones of the original domain triangle). However, it is easy to find bivariate polynomials where both degree elevation and regular subdivision of the original domain triangle fail to produce a convex control net. In the very special case of quadratic bivariate polynomials it has been observed by Prautzsch [15], that it is always possible to subdivide the original domain triangle so that the control net gets convex. The suitable (non–regular) partition of the domain triangle, however, depends on the given quadratic polynomial.

Consequently, convexity of the control net is a relatively strong convexity criterion. It will be shown later that it applies to only $\approx 50\%$ of the convex bivariate polynomials. By using quadratic convexity conditions (i.e., quadratic inequalities of the BB coefficients) it is possible to avoid these difficulties in the bivariate case, see e.g. [3, 6, 17]. However, it is much easier to use linear conditions in the context of surface construction; many tasks can then be formulated as optimization problems with linear constraints. Moreover, the generalization of the quadratic conditions to higher dimensions seems to be difficult.

Weaker linear conditions were developed by Lai [13], He [12] and recently by Carnicer, Floater and Peña [1]. The present article derives a general construction which leads to linear sufficient convexity conditions for multivariate polynomials. As the main new feature of the construction, the obtained conditions can be made as weak as desired, simply by increasing the number of inequalities. More precisely, it is possible to adapt the convexity conditions to any finite set of strongly convex polynomials. The results of Chang and Davis [2], Lai [13] and Carnicer et al. [1] appear as special cases.

2. Preliminaries

At first we introduce some notions concerning multivariate polynomials in Bernstein–Bézier representation over simplices, their blossoms, and their subdivision with respect to sub–simplices.

2.1. Bernstein–Bézier representation

In the sequel we will examine some convexity conditions for a multivariate polynomial $p : \mathbb{R}^s \to \mathbb{R}$, $x \mapsto p(x)$ with $x = (x_1, \ldots, x_s)^\top$, of total degree d. Let a basis s–simplex

$$\nu = \mathrm{hull}(v_0, \ldots, v_s) \tag{1}$$

with the $s+1$ affinely independent corners $v_0, \ldots, v_s \in \mathbb{R}^s$ be given. We denote with $\Lambda = \Lambda(\nu, x) = (\lambda_0, \ldots, \lambda_s)^\top$ the barycentric coordinates of a point $x \in \mathbb{R}^s$ with respect to ν,

$$x = \sum_{i=0}^{s} \lambda_i \, v_i \quad \text{and} \quad \sum_{i=0}^{s} \lambda_i = 1. \tag{2}$$

Adopting standard multi–index notations we write

$$|\beta| = \beta_0 + \ldots + \beta_s, \quad \beta! = \beta_0! \, \ldots \, \beta_s! \quad \text{and} \quad \Lambda^\beta = \lambda_0^{\beta_0} \, \ldots \, \lambda_s^{\beta_s} \tag{3}$$

for $\beta = (\beta_0, \ldots, \beta_s) \in \mathbb{Z}_+^{s+1}$ ($\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$). The Bernstein polynomials of degree d (with respect to the simplex ν) are then given by

$$B_\beta^d(\Lambda) = \frac{d!}{\beta!}\Lambda^\beta \text{ with } \beta \in \mathbb{Z}_+^{s+1}, |\beta| = d. \tag{4}$$

As the Bernstein polynomials form a basis of all multivariate polynomials of maximal degree d, the given polynomial $p(x)$ has a unique Bernstein–Bézier (BB) representation with respect to the simplex ν,

$$p(x) = \sum_{\beta \in \mathbb{Z}_+^{s+1}, |\beta| = d} b_\beta(p, \nu)\ B_\beta^d(\Lambda(\nu, x)). \tag{5}$$

The coefficients $b_\beta(p, \nu) \in \mathbb{R}$ are the BB coefficients of the polynomial p with respect to the basis simplex ν.

2.2. Multivariate blossoms

Suppose one wants to generate the BB representation of $p(x)$ with respect to another basis simplex

$$\omega = \text{hull}(w_0, \ldots, w_s) \tag{6}$$

with the $s+1$ affinely independent corners $w_0, \ldots, w_s \in \mathbb{R}^s$. The most compact description of the corresponding transformation of the coefficients $b_\beta(p, .)$ can be derived with the help of the blossom of the polynomial p. Before doing so we formulate some notions from [16] for the s–dimensional setting.

Take d copies x_i of the argument $x \in \mathbb{R}^s$ of the polynomial p, which correspond to d copies of the barycentric coordinates $\Lambda_i = (\lambda_{i,0}, \ldots, \lambda_{i,s})^\top$, $i = 1, \ldots, d$. Let $\Lambda_{i|k}^\beta$ be the k–th factor of the product Λ_i^β,

$$\Lambda_{i|k}^\beta = \lambda_{i,j} \ \Leftrightarrow\ \beta_0 + \ldots + \beta_{j-1} < k \leq \beta_0 + \ldots + \beta_j. \tag{7}$$

The blossom of a multivariate Bernstein polynomial $B_\beta^d(\Lambda)$ may be defined by

$$\text{Bloss}_\beta^d(\Lambda_1, \ldots, \Lambda_d) = \frac{1}{\beta!} \sum_{\pi \in \Pi^d} \prod_{i=1}^d \Lambda_{\pi(i)|i}^\beta \tag{8}$$

where Π^d is the set of all permutations π of $(1, \ldots, d)$. For instance, the blossom of $B_{0,2,1}^3(\Lambda) = \frac{3!}{0!\,2!\,1!}{\lambda_0}^0{\lambda_1}^2{\lambda_2}^1$ evaluates to

$$\begin{aligned}\text{Bloss}_{0,2,1}^3(\Lambda_1, \Lambda_2, \Lambda_3) = \ & \frac{1}{2!\,1!}(\lambda_{1,1}\lambda_{2,1}\lambda_{3,2} + \lambda_{1,1}\lambda_{3,1}\lambda_{2,2} + \lambda_{2,1}\lambda_{1,1}\lambda_{3,2} \\ & + \lambda_{2,1}\lambda_{3,1}\lambda_{1,2} + \lambda_{3,1}\lambda_{1,1}\lambda_{2,2} + \lambda_{3,1}\lambda_{2,1}\lambda_{1,2}).\end{aligned} \tag{9}$$

The blossom (also called the polar form) of the polynomial p is then obtained as the linear combination of the blossoms of the Bernstein polynomials,

$$\text{Bloss } p(x_1,\ldots,x_d) = \sum_{\beta\in\mathbb{Z}_+^{s+1},|\beta|=d} b_\beta(p,\nu)\,\text{Bloss}_\beta^d(\Lambda_1,\ldots,\Lambda_d) \tag{10}$$

with $\Lambda_i = \Lambda(\nu, x_i)$. It has (among others) the following properties (see [16]):

- It is a multi-affine mapping, i.e., linear in each of its d arguments,

$$\text{Bloss } p(\ldots, r\,x_i + s\,y_i, \ldots) = r\,\text{Bloss } p(\ldots, x_i, \ldots) + s\,\text{Bloss } p(\ldots, y_i, \ldots).$$

- It is symmetric with respect to arbitrary permutations of its arguments,

$$\text{Bloss } p(x_1,\ldots,x_d) = \text{Bloss } p(x_{\pi(1)},\ldots,x_{\pi(d)})$$

  for all permutations $\pi \in \Pi^d$.

- Restricting the blossom to the diagonal reproduces the polynomial p,

$$p(x) = \text{Bloss } p(x,\ldots,x).$$

- The blossom of $p(x)$ is unique, it does not depend on the choice of basis simplex ν.

With the help of the blossom we can compute the coefficients of the polynomial $p(x)$ with respect to the new basis simplex ω. The BB form of the polynomial p with respect to that simplex is given by

$$p(x) = \sum_{\beta\in\mathbb{Z}_+^{s+1},|\beta|=d} b_\beta(p,\omega)\,B_\beta^d(\Lambda(\omega,x)) \tag{11}$$

with the coefficients

$$b_\beta(p,\omega) = \text{Bloss } p(\underbrace{w_0,\ldots,w_0}_{\beta_0 \text{ times}}, \underbrace{w_1,\ldots,w_1}_{\beta_1 \text{ times}}, \ldots \underbrace{w_s,\ldots,w_s}_{\beta_s \text{ times}}). \tag{12}$$

The proof of this observation is a consequence of the multivariate version of the de Casteljau algorithm. Figure 1 gives a schematic illustration of Eq. (12). The coefficient $b_\beta(p,\omega)$ is attached to the point with barycentric coordinates β/d in ω.

The blossom of p depends continuously on its arguments, hence

$$\lim_{\substack{\mathrm{w}_i\to\mathrm{q} \\ (i=1,\ldots,s)}} b_\beta(p,\omega) = p(q). \tag{13}$$

Thus, if we restrict the polynomial p to simplices whose corners converge to a point $q \in \mathbb{R}^s$, then the coefficients of the BB representation converge to the value of p at this point q.

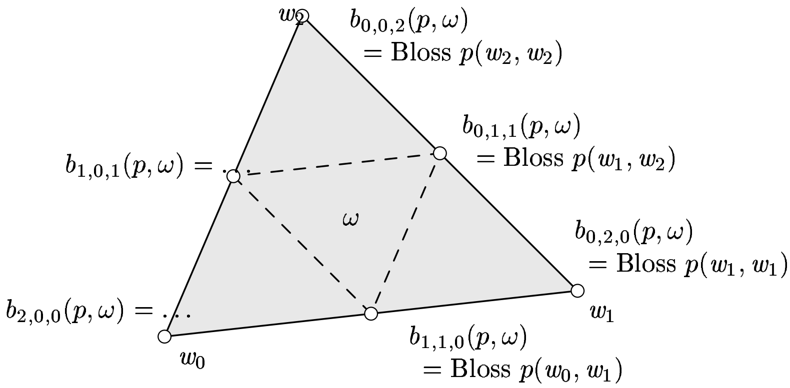

Figure 1: Computing the BB coefficients of a polynomial ($s = 2$, $d = 2$) via its blossom (scheme).

2.3. A triangulation of an s–simplex

For subdividing multivariate polynomials as described below we need to introduce a triangulation of an s–simplex. We will use the same canonical triangulation as in [5, p. 273]. Consider a permutation $\pi \in \Pi_s$ and a corner point $v \in \mathbb{Z}_+^s$. Let $\sigma_{\mathrm{v},\pi}$ be the simplex

$$\sigma_{\mathrm{v},\pi} = \mathrm{hull}(v, v{+}e_{\pi(1)}, v{+}e_{\pi(1)}{+}e_{\pi(2)}, \ldots, v{+}e_{\pi(1)}{+}e_{\pi(2)}{+}\ldots{+}e_{\pi(s)}) \tag{14}$$

where e_k is the k–th unit vector $(0,\ldots,0,1,0,\ldots,0)^\top \in \mathbb{R}^s$. We construct a triangulation (i.e., a decomposition into disjoint simplices) for the unit simplex $\Delta_k^{(s)}$ of size k in $\mathbb{R}^s$,

$$\Delta_k^{(s)} = \mathrm{hull}(0, k\,e_1, k\,e_1{+}k\,e_2, \ldots, k\,e_1{+}\ldots{+}k\,e_s). \tag{15}$$

According to [5, p. 273] such a triangulation may be obtained from

$$\mathcal{T}(\Delta_k^{(s)}) = \{\, \sigma_{\mathrm{v},\pi} \mid v \in \mathbb{Z}_+^s, \pi \in \Pi_s, \sigma_{\mathrm{v},\pi} \subseteq \Delta_k^{(s)} \,\}, \tag{16}$$

see Figure 2 for an illustration. The triangulation of an arbitrary simplex σ with the refinement level k is now obtained by an affine mapping which maps the corners of the unit simplex $\Delta_k^{(s)}$ onto the corners of σ. Note that this mapping is not uniquely determined. We denote by $\mathcal{T}_k(\sigma)$ the triangulation of σ which is obtained from *one* of these mappings (without referring to the choice of the mapping).

Consider again the polynomial p in BB representation with respect to the simplex ν. Based on its blossom we may compute its BB representation with respect to any simplex from $\mathcal{T}_k(\nu)$. We introduce the abbreviation

$$\mathcal{B}_k(p,\nu) = \{\, b_\beta(p,\tau) \mid \beta \in \mathbb{Z}_+^{s+1},\ |\beta| = d,\ \tau \in \mathcal{T}_k(\nu) \,\} \tag{17}$$

for the set of coefficients which is obtained by collecting the resulting coefficients. All these coefficients are certain constant affine combinations of the original coefficients $b_\beta(p,\nu)$.

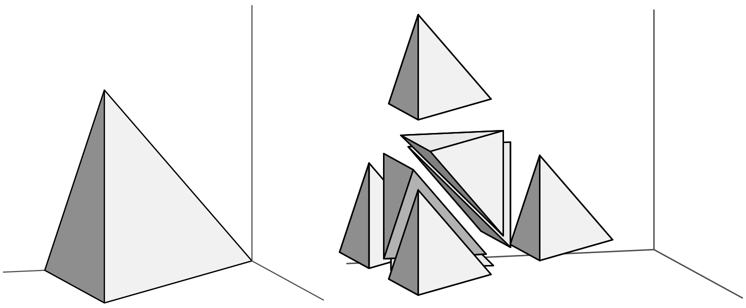

Figure 2: Canonical split $\mathcal{T}_2$ of the simplex $\Delta_1^{(2)}$ (left) into 8 tetrahedrons (right).

Resulting from (13), for increasing refinement level $k \to \infty$ the coefficients from (17) converge to values of the polynomial p on ν. As a consequence one has:

Lemma 1. *Consider a finite set of polynomials $p_i(x)$ in BB representation with respect to the s–simplices $\nu_i \subset \mathbb{R}^s$ $(i = 1, \ldots, P)$, where p_i is assumed to be strictly positive values for $x \in \nu_i$. If the refinement level k is chosen big enough, then all coefficients $\bigcup_{i=1}^{P} \mathcal{B}_k(p_i, \nu_i)$ of the BB representations with respect to the resulting sub-simplices $\mathcal{T}_{||}(\nu_\rangle)$ are nonnegative.*

3. Conditionally non–negative definite matrices

Linear constraints that guarantee that the Hessian matrix of a function is non–negative definite (or, more appropriate in the context of multivariate polynomials in BB representation, conditionally non–negative definite) are the essential ingredients for constructing linear convexity conditions. We present a general construction for such constraints.

3.1. Construction of the constraints

Consider a symmetric real $m{\times}m$–matrix $A = (a_{i,j})_{i,j=1,\ldots,m}$. This matrix is said to be conditionally non–negative definite (also called conditionally positive semidefinite, see [5]) if the inequality

$$c^\top A\ c = \sum_{i=1}^{m}\sum_{j=1}^{m} c_i\ c_j\ a_{i,j} \geq 0 \tag{18}$$

holds for all vectors $c = (c_1, \ldots, c_m)^\top \in \mathbb{R}^m$ satisfying the side–condition

$$c_1 + c_2 + \ldots + c_m = 0. \tag{19}$$

Eq. (19) describes a hyperplane through the origin in $\mathbb{R}^m$. We seek for linear inequalities of the components $a_{i,j}$ which imply this property of A. Such inequalities are obtained from the following general construction:

We choose a convex $m-1$–dimensional polyhedron which is contained within in the hyperplane (19) of $\mathbb{R}^m$. It facets must be $m-2$–dimensional simplices, and it has to possess the origin O as an interior point. We denote by $r_1, r_2, \ldots, r_V$ the vertices (which are to satisfy (19)!) and by $r_{f(r,0)}, r_{f(r,1)}, \ldots, r_{f(r,m-2)}$ the facets ($r = 1, \ldots, F$) of this polyhedron.

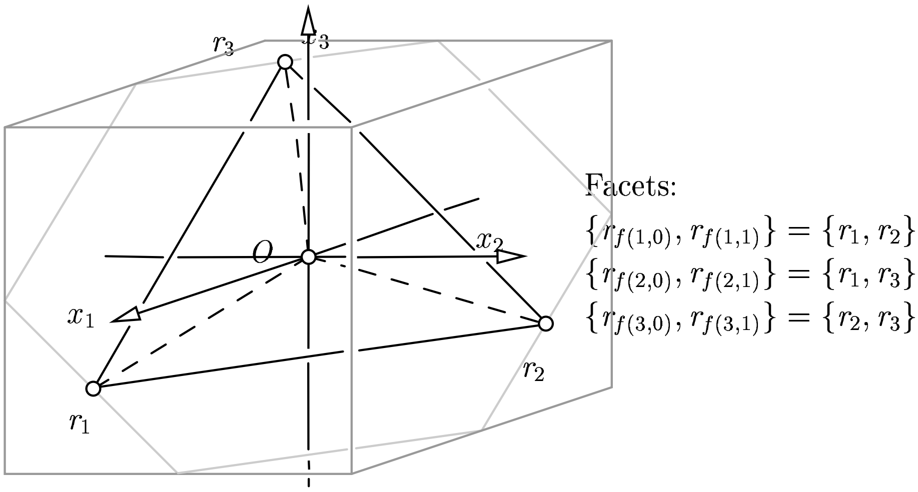

Figure 3: A polyhedron within the hyperplane (19) for $m = 3$.

As an example, Figure 3 shows the situation for $m = 3$. The hyperplane (19) intersects the cube with the corners $(\pm 1, \pm 1, \pm 1)$ in a regular hexagon. The polyhedron has been chosen as a regular triangle within this hyperplane.

The expression

$$f_r(y) = \sum_{i=0}^{m-2} \lambda_i \, r_{f(r,i)} \text{ with } y \in \Delta_1^{(m-2)} \subset \mathbb{R}^{m-2} \text{ and } \Lambda = \Lambda(y, \Delta_1^{(m-2)}) \tag{20}$$

is a linear parametric representation of the r–th facet with the parameter domain $\Delta_1^{(m-2)} \subset \mathbb{R}^{m-2}$. By restricting the quadratic form (18) to this facet we get the quadratic polynomial

$$q_r(y) = f_r(y)^\top A \, f_r(y) = \sum_{\beta \in \mathbb{Z}_+^{m-1}, |\beta|=2} b_\beta(q_r, \Delta_1^{(m-2)}) \;\; B_\beta^2(\Lambda(y, \Delta_1^{(m-2)})) \tag{21}$$

with the coefficients

$$b_{e_i+e_j}(q_r, \Delta_1^{(m-2)}) = r_{f(r,i)}^\top A \, r_{f(r,j)} \quad (i, j = 0, \ldots, m-2), \tag{22}$$

where e_i is the i–th unit vector of $\mathbb{Z}^{m-1}$. We chose a refinement level k and consider the inequalities

$$\mathcal{I}_k(A) = \bigcup_{r=1}^{F} \mathcal{B}_k(q_r, \Delta_1^{(m-2)}) \;\; \geq \;\; 0 \tag{23}$$

with the abbreviation

$$\{r, s, \dots, t\} \geq 0 \quad \Leftrightarrow \quad r \geq 0, s \geq 0, \dots, t \geq 0. \tag{24}$$

These inequalities depend linearly on the components $(a_{i,j})_{i,j=1,\dots,m}$ of the matrix A, as the BB coefficients (with respect to $\Delta_1^{(m-2)}$) of the polynomials $q_r(y)$ are given by (22) and increasing the refinement level k produces linear combinations of them.

Proposition 2. *If the components $(a_{i,j})_{i,j=1,\dots,m}$ satisfy the linear inequalities (23), then the matrix A is conditionally nonnegative definite.*

Proof. Consider an arbitrary point c within the hyperplane (19) of $\mathbb{R}^m$. According to the assumptions made about the polyhedron with the vertices $r_1, \dots, r_V$, the line $\mu\, c$ ($\mu \in \mathbb{R}$) intersects two of the facets of the polyhedron at least, thus

$$c = \frac{1}{\mu}\, f_r(y) \tag{25}$$

holds for certain $r \in \{1, .., F\}$, $y \in \Delta_1^{(m-2)} \subset \mathbb{R}^{m-2}$ and $\mu \in \mathbb{R} \setminus \{0\}$. Moreover, y is contained in one simplex $\sigma \in \mathcal{T}_k(\Delta_1^{(m-2)})$ of the triangulation of $\Delta_1^{(m-2)}$ with refinement level k. Hence,

$$c^\top A\ c = \frac{1}{\mu^2}\, f_r(y)^\top A\ f_r(y) = \frac{1}{\mu^2}\, q_r(y) = \sum_{\beta \in \mathbb{Z}_+^{m-1}, |\beta|=2} \underbrace{b_\beta(q_r, \sigma)}_{(*)}\ \underbrace{B_\beta^2(\Lambda(y, \sigma))}_{(**)} \geq 0. \tag{26}$$

The coefficients $(*)$ are nonnegative as the corresponding inequalities are contained in the set (23). The Bernstein polynomials $(**)$ are nonnegative as well, due to $y \in \sigma$. Thus, the inequality $c^\top A\ c \geq 0$ is fulfilled. □

Choosing different polyhedrons leads to different sets of inequalities. Two possible choices will be discussed in more detail:

Example 1: Cross–polyhedron. The $V = 2(m-1)$ vertices $r_i \in \mathbb{R}^m$ of the polyhedron in the hyperplane (19) are chosen as

$$\begin{aligned} r_1 = -r_2 = (\ 1, 0, 0, \dots, 0, -1\ )^\top, \quad r_3 = -r_4 = (\ 0, 1, 0, \dots, 0, -1\ )^\top, \\ \dots, \quad r_{V-1} = -r_V = (\ 0, 0, 0, \dots, 1, -1\ )^\top. \end{aligned} \tag{27}$$

The $F = 2^{m-1}$ facets of this polyhedron have lists of vertices from the set

$$\{\ (r_{f(r,0)}, r_{f(r,1)}, \dots, r_{f(r,m-2)})\ |\ r{=}1, \dots, F\ \} = \{r_1, r_2\} \times \{r_3, r_4\} \times \dots \times \{r_{V-1}, r_V\}. \tag{28}$$

For instance, in the case $m = 3$, the polyhedron is the image of a square within the plane $x_3 = 0$ under projection parallel to the x_3–axis into the plane (19). Parameterizing the facets and substituting them into the quadratic form leads to four quadratic polynomials. Resulting from the central symmetry of the polyhedron, only two of them are different. These polynomials are

$$q_{1/2}(y) = (a_{3,3}+a_{1,1}-2\,a_{3,1})\,B^2_{2,0}(\Lambda) \\ \pm\,(a_{3,1}-a_{3,3}+a_{3,2}-a_{2,1})\,B^2_{1,1}(\Lambda)+(a_{2,2}+a_{3,3}-2\,a_{3,2})\,B^2_{0,2}(\Lambda) \tag{29}$$

with parameter domain $y \in \Delta^{(1)}_1 \subset \mathbb{R}^1$.

Similarly, in the case $m = 4$, the polyhedron is the image of an octahedron within the hyperplane $x_4 = 0$ under parallel projection into the hyperplane (19). Parameterizing the facets now leads to eight quadratic polynomials, but only four of them are different. They may be obtained from

$$\begin{aligned} q_{1/2/3/4}(y) &= (a_{4,4}+a_{1,1}-2\,a_{4,1})\,B^2_{2,0,0}(\Lambda)+s_1\,(-a_{4,1}+a_{4,4}+a_{2,1}-a_{4,2})\,B^2_{1,1,0}(\Lambda) \\ &+s_2\,(-a_{4,1}-a_{4,3}+a_{4,4}+a_{3,1})\,B^2_{1,0,1}(\Lambda)+(a_{4,4}+a_{2,2}-2\,a_{4,2})\,B^2_{0,2,0}(\Lambda) \\ &+s_3\,(-a_{4,2}-a_{4,3}+a_{4,4}+a_{3,2})\,B^2_{0,1,1}(\Lambda)+(a_{3,3}+a_{4,4}-2\,a_{4,3})\,B^2_{0,0,2}(\Lambda) \\ &\text{with } (s_1,s_2,s_3) \in \{(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)\} \end{aligned} \tag{30}$$

and they have the parameter domain $y \in \Delta^{(2)}_1 \subset \mathbb{R}^2$. Based on their blossoms (which are obtained by replacing the quadratic Bernstein polynomials with their blossoms) the linear inequalities (23) can now easily be generated.

Resulting from the asymmetry of the polyhedron, the coefficients of the obtained quadratic polynomials are not symmetric with respect to simultaneous permutations of the lines and columns of the matrix (which keep its symmetry), and the obtained linear inequalities are non–symmetric too. However, as a major advantage of this polyhedron, it is easy to detect and to remove dependencies of the constraints. This is a more difficult task for the second polyhedron:

Example 2: Regular simplex. The $V = m$ vertices $r_i \in \mathbb{R}^m$ of the simplex in the hyperplane (19) are chosen as

$$r_1 = (\,m-1,-1,-1,\ldots,-1\,)^\top, \quad r_2 = (\,-1,m-1,-1,\ldots,-1\,)^\top, \\ \ldots, \quad r_m = (\,-1,-1,-1,\ldots,m-1\,)^\top. \tag{31}$$

The $F = m$ facets of this polyhedron have the lists of vertices

$$\{\,(r_{f(r,0)}, r_{f(r,1)}, \ldots, r_{f(r,m-2)}) \mid r{=}1,\ldots,F\,\} = \\ \{(r_1,\ldots,r_{m-2},r_{m-1}),(r_1,\ldots,r_{m-2},r_m),\ldots,(r_2,\ldots,r_{m-1},r_m)\}, \tag{32}$$

they result by choosing all possible subsets of $m-1$ vertices. For instance, in the case $m = 3$, the simplex is a regular triangle within the plane (19), see Figure 3. Parameterizing the three facets and substituting them into the quadratic form leads to the three quadratic polynomials

$$q_{1/2/3}(y) = c_{i,i}\, B^2_{2,0}(\Lambda) + c_{i,j}\, B^2_{1,1}(\Lambda) + c_{j,j}\, B^2_{0,2}(\Lambda) \tag{33}$$

with $(i,j) \in \{(1,2),(1,3),(2,3)\}$, and with the coefficients

$$\begin{aligned} c_{r,r} &= 4\, a_{r,r} - 4\, a_{s,r} - 4\, a_{t,r} + 2\, a_{t,s} + a_{s,s} + a_{t,t}, \\ c_{r,s} &= 5\, a_{s,r} - 2\, a_{r,r} - 2\, a_{s,s} + a_{t,t} - a_{t,r} - a_{t,s}, \end{aligned} \tag{34}$$

for $(r,s,t) \in \Pi^3$. These polynomials have the parameter domain $y \in \Delta_1^{(1)} \subset \mathbb{R}^1$.

Similarly, in the case $m = 4$, the simplex is a regular tetrahedron within the plane (19) and we get the four quadratic polynomials

$$\begin{aligned} q_{1/2/3/4}(y) = {} & c_{i,i} B^2_{2,0,0}(\Lambda) + c_{i,j} B^2_{1,1,0}(\Lambda) \\ & + c_{i,l} B^2_{1,0,1}(\Lambda) + c_{j,j} B^2_{0,2,0}(\Lambda) + c_{j,l} B^2_{0,1,1}(\Lambda) + c_{l,l} B^2_{0,0,2}(\Lambda), \end{aligned} \tag{35}$$

$(i,j,l) \in \{(1,2,3),(1,2,4),(1,3,4),(2,3,4)\}$, with the coefficients

$$\begin{aligned} c_{r,r} &= 9\, a_{r,r} - 6\, a_{s,r} - 6\, a_{t,r} - 6\, a_{u,r} + 2\, a_{t,s} + 2\, a_{u,s} + 2\, a_{u,t} + a_{s,s} + a_{t,t} + a_{u,u}, \\ c_{r,s} &= 20\, a_{s,r} - 6\, a_{r,r} - 6\, a_{s,s} + 4\, a_{u,t} - 4\, a_{t,r} - 4\, a_{t,s} - 4\, a_{u,r} - 4\, a_{u,s} + 2\, a_{t,t} + 2\, a_{u,u}, \end{aligned} \tag{36}$$

for $(r,s,t,u) \in \Pi^4$. These polynomials have the parameter domain $y \in \Delta_1^{(2)} \subset \mathbb{R}^2$.

With the help of the blossoms, the linear inequalities (23) can now easily be generated. Resulting from the symmetry of the polyhedron, the coefficients of the obtained quadratic polynomials are symmetric with respect to permutations of the lines and columns of the matrix. Hence, for $m \leq 4$ we obtain linear inequalities which are symmetric as well, as also the triangulations $\mathcal{T}_k(\sigma)$ of $m-2$–simplices (line segments or triangles) are symmetric. However, this is no longer true for $m > 4$.

For constructing the polynomials $q_r(y)$ it would be sufficient to use a polyhedron which covers half of the unit sphere only, but this would destroy the symmetry of the set of polynomials $q_r(y)$ for $m > 3$. On the other hand, using a sphere–like polyhedron we get conditions which contain dependencies, and for certain applications it may therefore be necessary to detect and to eliminate the dependent constraints. For instance, such dependencies can be found with the help of the simplex algorithm.

3.2. Asymptotic necessity

As the main new feature of the above construction, the linear constraints can be made as weak as desired, by increasing the refinement level k.

Proposition 3. *Consider an arbitrary finite set* $A^{(l)} = (a_{i,j}^{(l)})_{i,j=1,\ldots,m}$ *of conditionally positive definite symmetric* $m{\times}m$*–matrices* $(l = 1, \ldots, R)$*, i.e.,*

$$c^\top A^{(l)} c = \sum_{i=1}^{m} \sum_{j=1}^{m} c_i\, c_j\, a_{i,j}^{(l)} > 0 \qquad (l = 1, \ldots, R) \tag{37}$$

holds for all vectors $c = (c_0, \ldots, c_m)^\top \neq (0, 0, \ldots, 0)$ *which satisfy (19). If the refinement level* k *is big enough, then the components of all matrices fulfill the linear inequalities* $\mathcal{I}_k(A^{(l)}) \geq 0$*, cf. (23).*

Proof. Parameterizing the facets of the polyhedron with vertices $r_0, \ldots, r_V$ and substituting them into the quadratic forms (37) leads to finitely many nonnegative polynomials $q_r^{(l)}(y)$, where the upper index refers to the matrix. These polynomials are even positive for $y \in \Delta_1^{m-2}$, as all matrices were assumed to be conditionally positive definite. Hence, it is possible to apply Lemma 1. If the refinement level k is big enough, then all Bézier coefficients $\mathcal{B}_k(q_r^{(l)}, y \in \Delta_1^{m-2})$ are nonnegative. Then, all matrices fulfill the inequalities $\mathcal{I}_k(A^{(l)}) \geq 0$. □

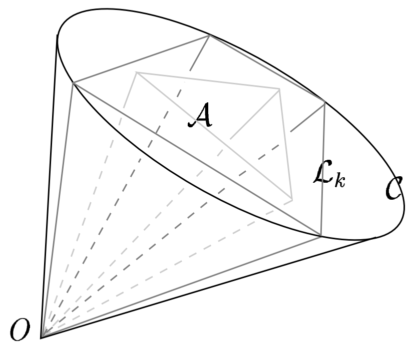

Figure 4: Geometric interpretation of Proposition 3 for $m = 2$.

This result can be interpreted in a geometric way, see Figure 4. The conditionally nonnegative definite symmetric $m{\times}m$–matrices form a convex cone $\mathcal{C}$ within an $\binom{m+1}{2}$–dimensional (due to the number of different components) linear space. The matrices which satisfy the constraints (23) form another convex cone $\mathcal{L}_k$ which is bounded by hyperplanes through the origin. This cone is inscribed to $\mathcal{C}$. The finitely many conditionally positive definite matrices $A^{(l)}$ span a third convex cone $\mathcal{A}$ which has no points on the boundary of $\mathcal{C}$, except for the origin O. We can always find a refinement level k such that $\mathcal{A}$ is contained within $\mathcal{L}_k$.

3.3. Comparison with other linear conditions

We compare our conditions for the case $m = 3$ (which corresponds to convexity of bivariate polynomials) with other linear conditions.

Chang and Davis [2] derived the conditions

$$\mathcal{CD} = \{a_{3,3}+a_{1,2}-a_{1,3}-a_{2,3},\ a_{1,1}+a_{2,3}-a_{1,2}-a_{1,3},\ a_{2,2}+a_{1,3}-a_{2,3}-a_{1,2}\} \geq 0 \qquad (38)$$

which were also discussed by Dahmen and Micchelli [5] in the multivariate setting. These inequalities are equivalent to convexity of the control net of the corresponding triangular Bézier patch. Based on diagonal dominance of matrices, Lai [13] developed the conditions

$$\begin{aligned}\mathcal{L} = \ & \{2\,a_{3,3}+a_{2,2}+a_{1,2}-a_{1,3}-3\,a_{2,3},\ 2\,a_{3,3}+a_{1,1}+a_{1,2}-a_{2,3}-3\,a_{1,3},\\ & a_{1,1}+a_{2,3}-a_{1,2}-a_{1,3},\ a_{2,2}+a_{1,3}-a_{1,2}-a_{2,3}\} \quad \geq 0\end{aligned} \qquad (39)$$

Using the idea of weaker diagonal dominance, Carnicer, Floater and Peña [1] recently derived the sufficient conditions

$$\begin{aligned}\mathcal{CFP} = \{ \ & 2\,a_{1,1}+a_{2,2}+a_{2,3}-3\,a_{1,2}-a_{1,3},\ 2\,a_{1,1}+a_{3,3}+a_{2,3}-3\,a_{1,3}-a_{1,2},\\ & 2\,a_{2,2}+a_{1,1}+a_{1,3}-3\,a_{1,2}-a_{2,3},\ 2\,a_{2,2}+a_{3,3}+a_{1,3}-3\,a_{2,3}-a_{1,2},\\ & 2\,a_{3,3}+a_{1,1}+a_{1,2}-3\,a_{1,3}-a_{2,3},\ 2\,a_{3,3}+a_{2,2}+a_{1,2}-3\,a_{2,3}-a_{1,3} \quad \} \geq 0.\end{aligned} \qquad (40)$$

The linear conditions which are obtained from (29), by choosing a cross–polyhedron, for the refinement level $k \in \{2, 3, .., 9\}$ (see (23) and (17) are denoted by $\mathcal{C}_k$. Similarly, the inequalities which are obtained from (33), by choosing a simplex, for the refinement level $k \in \{3, 4, .., 10, 12\}$ are denoted by $\mathcal{S}_k$. The considered refinement levels lead to constraints consisting of 18 independent inequalities at most.

It turns out that the above conditions (38), (39), (40) may be obtained as special cases of our construction:

$$\mathcal{CD} \geq 0 \ \Leftrightarrow\ \mathcal{S}_3 \geq 0, \quad \mathcal{L} \geq 0 \ \Leftrightarrow\ \mathcal{C}_2 \geq 0, \ \text{ and } \ \mathcal{CFP} \geq 0 \ \Leftrightarrow\ \mathcal{S}_6 \geq 0. \qquad (41)$$

In order to compare the various conditions we randomly generated 100.000 symmetric 3×3 matrices A with components from $[-1, 1]$. All conditions are homogeneous; they depend only on the ratios of the matrix components, i.e., on the direction of the vector $\vec{a} = (a_{1,1}, a_{1,2}, a_{1,3}, a_{2,2}, a_{2,3}, a_{3,3}) \in \mathbb{R}^6$. For our experiment we therefore considered only those of the generated matrices which possess components within the unit sphere of $\mathbb{R}^6$,

$$a_{1,1}^2 + a_{1,2}^2 + a_{1,3}^2 + a_{2,2}^2 + a_{2,3}^2 + a_{3,3}^2 \leq 1 \qquad (42)$$

This led to a uniform distribution of the directions $\vec{a}$. Only 1009 of the generated matrices were conditionally nonnegative definite and satisfied (42). The table below shows the

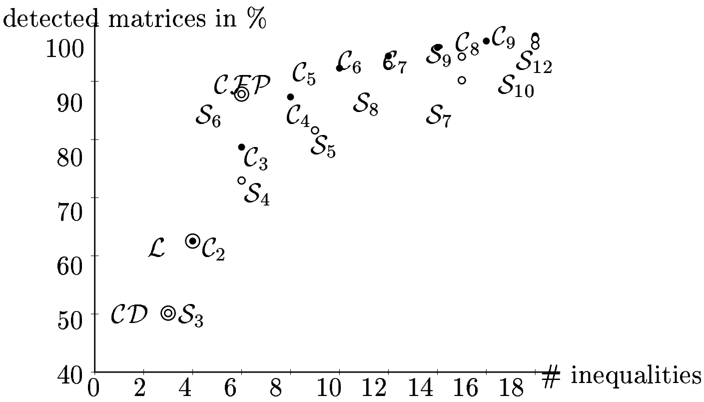

Figure 5: Comparison of convexity conditions

number and percentage of matrices that were detected by the different conditions and the number of linearly independent inequalities.

	$\mathcal{L}, \mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{CD}, \mathcal{S}_3$	$\mathcal{S}_4$	$\mathcal{S}_5$	$\mathcal{CFP}, \mathcal{S}_6$	$\mathcal{S}_7$	$\mathcal{S}_8$	$\mathcal{S}_9$	$\mathcal{S}_{10}$	$\mathcal{S}_{12}$
detected:	631	794	881	931	952	967	978	987	506	736	823	886	910	937	951	970	982
percentage:	62.5	78.7	87.3	92.3	94.4	95.8	96.9	97.8	50.1	72.9	81.6	87.8	90.2	92.9	94.3	96.1	97.3
# inequ.:	4	6	8	10	12	14	16	18	3	6	9	6	15	12	15	18	18

The relation between the number of inequalities and the percentage of detected constraints is depicted by Figure 5. The black and white dots indicate the conditions $\mathcal{C}_i$ and $\mathcal{S}_i$, respectively, whereas the conditions $\mathcal{CD}, \mathcal{L}, \mathcal{CFP}$ are represented by big circles. Note that the number of independent inequalities for the conditions $\mathcal{S}_i$ is not monotonic. This is due to the lack of central symmetry of the generating polyhedron. The figure illustrates the fact that the percentage of detected constraints tends to 100% if the refinement level k is increased.

4. Convex multivariate polynomials

With the help of the results of the previous section we derive linear sufficient convexity conditions for multivariate polynomials. The obtained conditions are shown to be asymptotically necessary.

4.1. Linear convexity conditions

Consider again the multivariate polynomial $p(x) : \mathbb{R}^s \to \mathbb{R}$, see (5), in BB representation with respect to the simplex $\nu \subset \mathbb{R}^s$. We associate with p and with the simplex ν the polynomials

$$s_\gamma(x) = \sum_{\beta \in \mathbb{Z}_+^{s+1}, |\beta| = d-2} b_\beta(s_\gamma, \nu)\, B_\beta^{d-2}(\Lambda(\nu, x)) \text{ with } b_\beta(s_\gamma, \nu) = b_{\beta+\gamma}(p, \nu) \tag{43}$$

for all $\gamma \in \mathbb{Z}_+^{s+1}$, $|\gamma| = 2$. These polynomials are related to the second partial derivatives of the BB representation (5) with respect to the barycentric coordinates $\Lambda(\nu, x)$,

$$s_{e_i+e_j} = d\,(d-1)\,\frac{\partial}{\partial\lambda_i}\,\frac{\partial}{\partial\lambda_j}\,s(\Lambda(\nu,x)) \quad \text{with} \quad s(\Lambda(\nu,x)) = p(x). \tag{44}$$

The following convexity criterion has firstly been formulated by Chang and Davis [2] for $s = 2$. It has later been generalized to arbitrary dimension by Dahmen and Micchelli [5].

Proposition 4. *The polynomial $p(x)$ is convex for $x \in \nu$ if and only if the $(s+1) \times (s+1)$ matrix $H(x)$ with the components*

$$h_{i,j}(x) = s_{e_i+e_j}(x) \quad i,j = 1,..,s+1 \tag{45}$$

is conditionally non–negative definite for all $x \in \nu$.

For the proof we refer to [2, 5]. In addition it can be shown that the polynomial is strongly convex if and only if the matrix (45) is conditionally positive definite.

We choose a refinement level $l = 1, 2, ..$ for the polynomials $s_\gamma(x)$ and consider the following set of matrices:

$$\begin{aligned}\mathcal{A}_l(p,\nu) = \{\ A = (a_{i,j})_{i,j=1,..,s+1} \quad \text{with} \quad & a_{i,j} = b_\beta(s_{e_i+e_j}, \tau)\ | \\ & |\beta| = d-2,\ \beta \in \mathbb{Z}_+^{s+1} \quad \text{and} \quad \tau \in \mathcal{T}_l(\nu)\ \}\end{aligned} \tag{46}$$

For generating these matrices one firstly has to subdivide the polynomials $s_\gamma(x)$ with respect to the triangulation $\mathcal{T}_l(\nu)$ of ν. The matrices result by collecting corresponding BB coefficients. For each individual simplex $\nu_0 \in \mathcal{T}_l(\nu)$ of the triangulation, the components of the corresponding matrices are taken from all sub–simplices with edge–length 2 of the BB coefficient scheme, see Figure 6 for an illustration of the bivariate case.

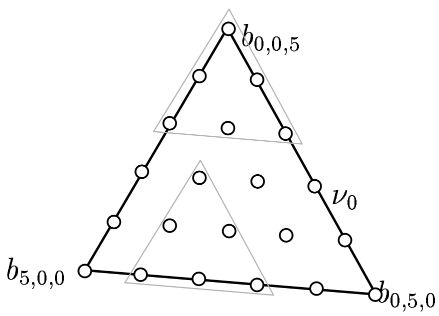

Figure 6: Components of matrices $A_l(p, \nu)$ in the bivariate case.

Note that subdividing the polynomials $s_\gamma(x)$ only weakens the resulting constraints if the polynomial $p(x)$ has degree $d \geq 4$. Otherwise the polynomials $s_\gamma(x)$ are constant or linear and their original BB coefficients are kept by subdividing them. Hence, if $d < 4$ holds one may always set $l = 1$.

Now we consider the inequalities

$$\mathcal{J}_{k,l}(p,\nu) \geq 0 \quad \text{with} \quad \mathcal{J}_{k,l}(p,\nu) = \bigcup \{\, \mathcal{I}_k(A) \mid A \in \mathcal{A}_l(p,\nu) \,\} \tag{47}$$

where $\mathcal{I}_k(A)$ has been introduced in (23) (after choosing a polyhedron within the hyperplane (19) and a refinement level k for its facets). These are linear inequalities for the BB coefficients $b_\beta(p,\nu)$.

Theorem 5. *If the inequalities (47) for the BB coefficients $b_\beta(p,\nu)$ hold, then the polynomial $p(x)$ is convex for $x \in \nu$.*

Proof. Consider the matrix (45) at an arbitrary point $x \in \nu$. Let $x \in \tau \in \mathcal{T}_l(\nu)$. Then

$$h_{i,j}(x) = \sum_{\beta \in \mathbb{Z}_+^{s+1}, |\beta| = d-2} b_\beta(s_{e_i+e_j}, \tau) \underbrace{B_\beta^{d-2}(\Lambda(\tau,x))}_{\geq 0} \text{ for } i,j = 1,..,s+1. \tag{48}$$

Hence, the matrix H is a non–negative linear combination of matrices from the set $\mathcal{A}_l(p,\nu)$. These matrices are conditionally non–negative definite as their components fulfill the inequalities (23). Thus, the Hessian $H(x)$ is conditionally nonnegative definite. □

For instance, choosing the refinement level $l = 1$ and the conditions $\mathcal{S}_3$ from the preceding section we obtain in the case $s = 2$ the convexity criterion of Chang and Davis which is equivalent to the convexity of the piecewise linear interpolant of the BB coefficients [2]. In the case of a cubic polynomial we obtain a system of 9 linear inequalities. Similarly, applying the conditions $\mathcal{S}_6$ (which are equivalent to $\mathcal{CFP}$) to a cubic polynomial $p(x)$ results in a system of 18 linear inequalities. of 18 linear inequalities.

4.2. Asymptotic necessity

The following observation is similar to Lemma 1.

Lemma 6. *Consider a finite set of polynomials $p_i(x)$ in BB representation with respect to the s–simplices $\nu_i \in \mathbb{R}^s$ $(i = 1,\dots,P)$, where $p_i(x)$ is assumed to be strongly convex for $x \in \nu_i$, i.e., the matrices (45) are conditionally positive definite. If the refinement level l is big enough, then all matrices $\mathcal{A}_l(p_i,\nu_i)$ are conditionally positive definite.*

The proof is a consequence of the subdivision property (13). Its details are omitted. If the refinement level l is increased, then the components of the matrices from $\mathcal{A}_l(p_i,\nu_i)$ converge to the values of the polynomials $s_\gamma(x)$, hence the matrices themselves converge to the Hessian matrices (45) which are assumed to be conditionally positive definite.

Theorem 7. *Consider again the set of polynomials from the previous Lemma. If the refinement levels k, l are big enough, then the BB coefficients of all polynomials fulfill the linear inequalities* $\mathcal{J}_{k,l}(p_i, \nu_i) \geq 0$.

The proof is an immediate consequence from the previous Lemma and from Proposition 3. By increasing the refinement levels, the linear sufficient convexity conditions $\mathcal{J}_{k,l}(p_i, \nu_i) \geq 0$ can be made as weak as desired. This result can be interpreted in a geometric way, analogous to Figure 4. Similar to the components of conditionally positive definite matrices, the BB coefficients of convex polynomials form a convex cone. In fact, for quadratic polynomials the components of the single matrix in $\mathcal{A}_1(p, \nu)$ are identical with the BB coefficients $b_\beta(p, \nu)$. Our construction inscribes another convex cone which is bounded by hyperplanes. If the refinement levels are increased simultaneously, then the inscribed cone converges to the cone of convex polynomials.

5. Concluding remarks

This article has presented a general construction for arbitrarily weak linear sufficient convexity conditions for multivariate polynomials. Other linear conditions appear as special cases of the general construction. With the help of the linearized convexity conditions, the construction or modification of convex multivariate polynomials (or piecewise polynomials) can now be formulated as optimization problems with linear constraints. For instance, Willemans and Dierckx have derived a method for convex surface fitting using piecewise quadratic polynomials on Powell–Sabin–splits [18]; they are led to an optimization problem with both linear and quadratic inequality constraints which is solved using nonlinear programming. Using the linearized convexity constraints it is now possible for formulate the same problem as a so–called quadratic programming problem (minimization of a quadratic objective function subject to linear constraints) which is one of the standard problem in optimization theory [7]. With the help of adaptive subdivision of the parameter domain it is possible to adapt the linearized convexity conditions to the specific situation. Also, the linearized convexity conditions can easily be applied to piecewise polynomials of higher dimension, higher degree, and higher order of differentiability.

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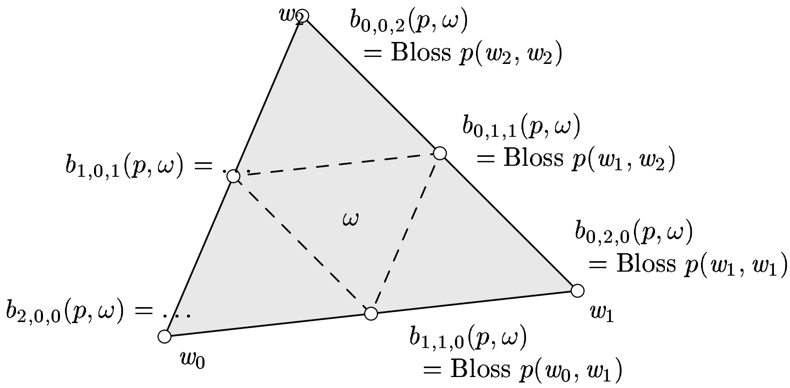

Figure 1: Computing the BB coefficients of a polynomial ($s = 2$, $d = 2$) via its blossom (scheme).

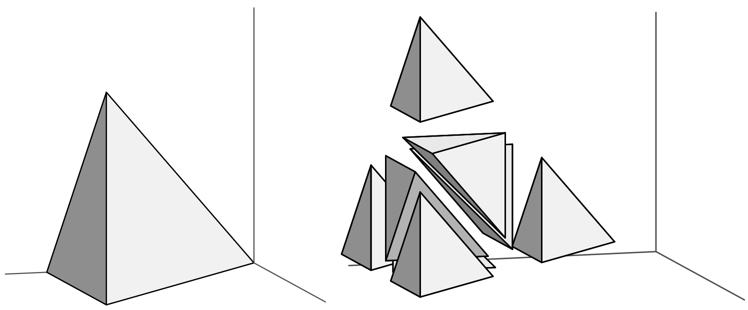

Figure 2: Canonical split $\mathcal{T}_2$ of the simplex $\Delta_1^{(2)}$ (left) into 8 tetrahedrons (right).

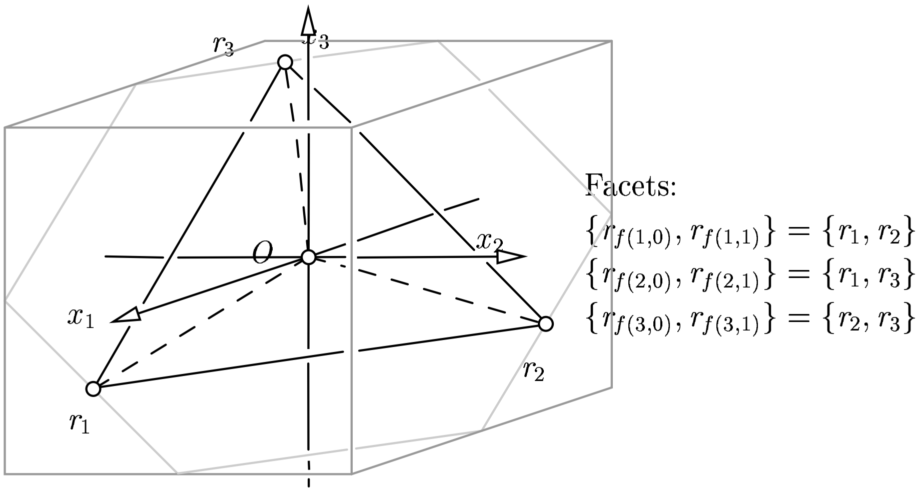

Figure 3: A polyhedron within the hyperplane (19) for $m = 3$.

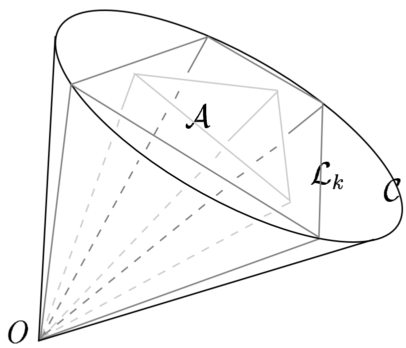

Figure 4: Geometric interpretation of Proposition 3 for $m = 2$.

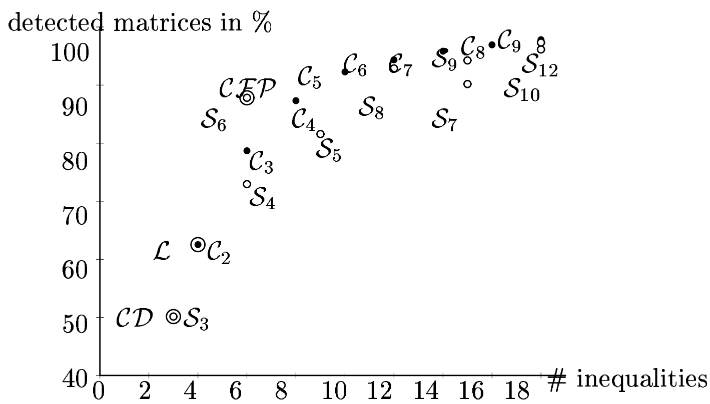

Figure 5: Comparison of convexity conditions

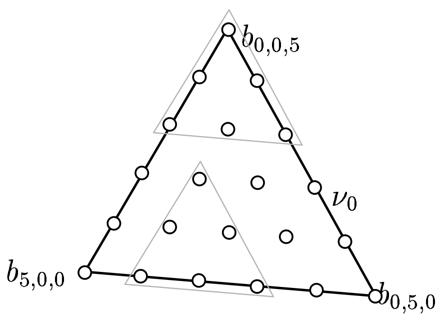

Figure 6: Components of matrices $A_l(p, \nu)$ in the bivariate case.