Bounding the Hausdorff Distance of Implicitly Defined and/or Parametric Curves

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Abstract. This paper is devoted to computational techniques for generating upper bounds on the Hausdorff distance between two planar curves. The results are suitable for pairs of implicitly defined and/or parametric curves. The bounds are computed directly from the control points resp. spline coefficients of the curves. They improve an earlier result of Sederberg [10, Eq. (6.2)]. Potential applications include error bounds for the approximate implicitization of spline curves, for the approximate parameterization of (piecewise) algebraic curves, and for algebraic curve fitting.

§1. Introduction

The notion of distance between two curves (and surfaces) is important for various applications of geometric design; see Bogacki and Weinstein [2] for a detailed discussion of several possible definitions. Parametric distance measures, such as the maximum norm of the difference vector of the parametric representations, are certainly useful in applications. These measures, however, are non–geometrical; they also tend to overestimate the real distance. Moreover, these measures cannot be used if one or both curves are given by an implicit representation, such as for piecewise algebraic spline curves, see [10].

This paper focuses on the well–known Hausdorff distance between two curves. We introduce the auxiliary notion of the footpoint distance, which is closely related to it, and develop a computational technique for generating upper bounds, directly from the control points resp. spline coefficients of the curves. The results are suitable for pairs of implicitly defined and/or parametric curves. They improve an earlier result of Sederberg [10], see end of Section 3.

The potential applications include error bounds for the approximate implicitization of spline curves and surfaces (cf. [4]), for the approximate parameterization of algebraic curves and surfaces (cf. [1]), and for curve and surface fitting with algebraic spline curves and surfaces.

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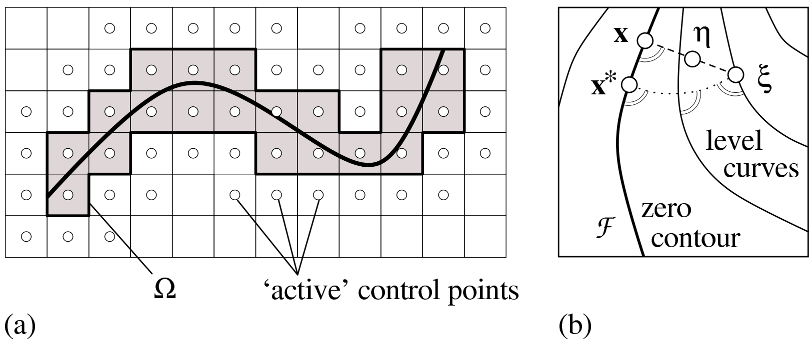

Fig. 1. (a) A planar curve can be defined as the zero contour of a biquadratic tensor–product spline function. (b) Footpoint $\mathbf{x} \in \mathcal{F}$ of a point $\boldsymbol{\xi}$ and the path of steepest descent (dotted line). See Lemma 1 and Corollary 2.

§2. Implicitly Defined Algebraic Spline Curves

Consider a closed bounded set $\Omega \subset \mathbb{R}^2$. The zero contour of a bivariate function $f(\mathbf{x}) = f(x_1, x_2) \not\equiv 0$ with the domain Ω,

$$\mathcal{F} = \{\, \mathbf{x} \mid f(\mathbf{x}) = 0 \;\wedge\; \mathbf{x} \in \Omega \,\}, \tag{1}$$

defines a planar curve $\mathcal{F}$, possibly consisting of more than one segment. If the function f is a bivariate spline function, then the curve $\mathcal{F}$ will be a piecewise algebraic curve, i.e., an algebraic spline curve. For instance, one may choose the function f as a tensor–product spline function of degree (d, d),

$$f(\mathbf{x}) = f(x_1, x_2) = \sum_{(i,j)\in\mathcal{I}} M_i^d(x_1)\, N_j^d(x_2)\, p_{i,j}, \qquad \mathbf{x} \in \Omega, \tag{2}$$

where the B-splines $M_i^d(x_1)$ and $N_j^d(x_2)$ are defined over suitable knot sequences. In this situation, the spline coefficients (control points) $p_{i,j} \in \mathbb{R}$ can be associated with the rectangular grid in the x_1x_2–plane, which is obtained from the Greville abscissas of the knot sequences, see e.g. [9]. The index set $\mathcal{I}$ contains the indices of all 'active' control points,

$$(i,j) \in \mathcal{I} \iff \exists\, \mathbf{x}^* = (x_1^*, x_2^*) \in \Omega : M_i^d(x_1^*)\, N_j^d(x_2^*) \neq 0. \tag{3}$$

An example is shown in Fig. 1a. The curve $\mathcal{F}$ is defined as the zero contour of a biquadratic tensor-product spline function with a regular grid of knot lines. Then, in the biquadratic case, the control points are associated with the centers of the cells. The domain Ω of the spline function consists of all grey cells. The active control points are marked by circles.

Alternatively, algebraic spline curves can be defined by piecing together the zero contours of triangular Bézier surface patches, see e.g. [10]. This representation is particularly useful if the implicit representation of the curve is generated by implicitizing a parametric one.

Throughout this paper we assume that the function f is at least C^1, and that its gradient satisfies $\nabla f(\mathbf{x}) \neq (0,0)$ for all $\mathbf{x} \in \Omega$. Consequently, the zero contour (1) is differentiable and has no singularities, such as multiple points or cusps.

§3. Distance of a Point

Consider a point $\boldsymbol{\xi} \in \Omega$. Any point $\mathbf{x} \in \mathcal{F}$ satisfying $(\boldsymbol{\xi} - \mathbf{x}) \times \nabla f(\mathbf{x}) = 0$ will be called a footpoint of $\boldsymbol{\xi}$, see Fig. 1b. Here, $\times$ is the two–dimensional exterior product, $\mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1$. The inner product of vectors will be denoted with $\mathbf{v} \cdot \mathbf{w}$, and the Euclidean distance of the corresponding points with $\operatorname{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$.

Lemma 1. *Consider a point $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega$. Let $\mathbf{x} = (x_1, x_2) \in \mathcal{F}$ be a footpoint of $\boldsymbol{\xi}$. If the line segment $\overline{\mathbf{x}\boldsymbol{\xi}}$ is contained in Ω, then the distance of the points $\boldsymbol{\xi}$ and $\mathbf{x}$ can be expressed as*

$$\operatorname{dist}(\mathbf{x}, \boldsymbol{\xi}) = \frac{\|\nabla f(\mathbf{x})\|}{|\nabla f(\mathbf{x}) \cdot \nabla f(\boldsymbol{\eta})|} \, |f(\boldsymbol{\xi})|, \tag{4}$$

where $\boldsymbol{\eta}$ is a certain point on the line segment connecting the points $\mathbf{x}$ and $\boldsymbol{\xi}$, cf. Fig. 1b.

Proof: Let $g(t)$ be the restriction of the function f to the normal of $\mathcal{F}$ at $\mathbf{x}$,

$$g(t) = f(\mathbf{x} + t \, \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}). \tag{5}$$

As $\mathbf{x} \in \mathcal{F}$ is a footpoint of $\boldsymbol{\xi}$, this function satisfies one of the equations $g(\operatorname{dist}(\mathbf{x}, \boldsymbol{\xi})) = f(\boldsymbol{\xi})$ or $g(-\operatorname{dist}(\mathbf{x}, \boldsymbol{\xi})) = f(\boldsymbol{\xi})$, depending on the orientation of the gradients. Moreover, $g(0) = 0$. Using the mean value theorem, we obtain in the first case

$$\frac{f(\boldsymbol{\xi}) - 0}{\operatorname{dist}(\mathbf{x}, \boldsymbol{\xi})} = g'(\lambda) = \nabla f(\boldsymbol{\eta}) \cdot \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|} \tag{6}$$

for some $\lambda \in [0, \operatorname{dist}(\mathbf{x}, \boldsymbol{\xi})]$, and with $\boldsymbol{\eta} = \mathbf{x} + \lambda \, \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}$. The assertion follows by solving this equation for $\operatorname{dist}(\mathbf{x}, \boldsymbol{\xi})$. The second case can be dealt with analogously. $\square$

This lemma can be used for bounding the distance of points $\boldsymbol{\xi} \in \Omega$ from its footpoints. With the help of the control points of the spline functions, we are able to generate an upper bound C on the length of the gradients,

$$\|\nabla f(\mathbf{x})\| \leq C \quad \text{for} \quad \mathbf{x} \in \Omega. \tag{7}$$

In addition, we may generate a lower bound D_h on the inner product of the gradients of any two neighbouring points whose distance does not exceed a certain constant h,

$$\|\nabla f(\mathbf{x}) \cdot \nabla f(\mathbf{y})\| \geq D_h \;\text{ holds for all }\; \mathbf{x}, \mathbf{y} \in \Omega \;\text{ with }\; \operatorname{dist}(\mathbf{x}, \mathbf{y}) \leq h. \tag{8}$$

The methods used for computing these bounds are described in Section 6. As a consequence of Lemma 1 we obtain the following result, which bounds the distance *without* computing the footpoint $\mathbf{x}$ of $\boldsymbol{\xi}$.

Corollary 2. *Consider again the situation of Lemma 1, and assume (7), (8). Let the parameter h be chosen such that $h \geq C/D_h\, f(\boldsymbol{\xi})$. The distance of the point $\boldsymbol{\xi}$ from its footpoint $\mathbf{x}$ on the curve $\mathcal{F}$ is then bounded by*

$$\operatorname{dist}(\mathbf{x}, \boldsymbol{\xi}) \leq \frac{C}{D_h}\, f(\boldsymbol{\xi}). \tag{9}$$

This result (and similarly the Theorems 3 and 4) can be used only if the parameter h is not too small. On the other hand, the smaller the parameter h, the bigger the lower bound D_h. The choice of a suitable constant h is addressed in Section 7.

If the point $\boldsymbol{\xi}$ approaches its footpoint $\mathbf{x}$, the bound (9) converges to zero, as $\boldsymbol{\xi} \to \mathbf{x}$ implies $f(\boldsymbol{\xi}) \to 0$.

This corollary improves an erroneous result in [10]. According to inequality (6.2) of [10], the distance can be bounded by $C\, f(\boldsymbol{\xi})$. (The original formula is slightly different, but it is in fact equivalent to this one.) However, this formula is valid only if additional assumptions about the domain Ω are satisfied, which have so far not been specified. More precisely, the domain Ω has to contain both the point $\boldsymbol{\xi}$ and the point $\mathbf{x}^*$ on the curve $\mathcal{F}$ which is obtained by following the path of steepest descent, starting at $\boldsymbol{\xi}$, see Fig. 1b. Generally, the latter point is *not* the footpoint $\boldsymbol{\xi}$.

§4. Distance of Two Implicitly Defined Curves

Let $\mathcal{F}$ be a planar curve defined by (1) and (2), and let the second curve $\mathcal{G}$ be defined by

$$\mathcal{G} = \{\, \mathbf{x} \mid g(\mathbf{x}) = 0 \;\wedge\; \mathbf{g} \in \Omega \,\}, \tag{10}$$

where $g(\mathbf{x})$ is a another bivariate spline function

$$g(\mathbf{x}) = g(x_1, x_2) = \sum_{(i,j)\in\mathcal{I}} M_i^d(x_1)\, N_j^d(x_2)\, q_{i,j}, \qquad \mathbf{x} \in \Omega, \tag{11}$$

with control points $q_{i,j} \in \mathbb{R}$ and domain Ω. The knots of $f(\mathbf{x})$ and $g(\mathbf{x})$ are assumed to be identical. Let $\mathcal{G}_0$ be the segment of $\mathcal{G}$ which consists of all points which have at least one footpoint on $\mathcal{F}$,

$$\mathcal{G}_0 = \{\mathbf{y} \mid \mathbf{y} \in \mathcal{G} \vee \exists \mathbf{x} \in \mathcal{F} : (\mathbf{y} - \mathbf{x}) \cdot \nabla f(\mathbf{x}) = 0\}, \tag{12}$$

see Fig. 2. We consider the maximum distance of the points of G_0 from their footpoints,

$$\operatorname{dist}^{\mathrm{F}}(\mathcal{F}, \mathcal{G}) = \sup_{\mathbf{y}\in\mathcal{G}_0} \; \inf_{\substack{\mathbf{x}\in\mathcal{F} \text{ is a} \\ \text{footpoint of } \mathbf{y}}} \operatorname{dist}(\mathbf{x}, \mathbf{y}). \tag{13}$$

This measure will be called the one–sided footpoint distance. By symmetrizing it we obtain the footpoint distance

$$\operatorname{Dist}^{\mathrm{F}}(\mathcal{F}, \mathcal{G}) = \max\{\, \operatorname{dist}^{\mathrm{F}}(\mathcal{F}, \mathcal{G}),\; \operatorname{dist}^{\mathrm{F}}(\mathcal{G}, \mathcal{F}) \,\}, \tag{14}$$

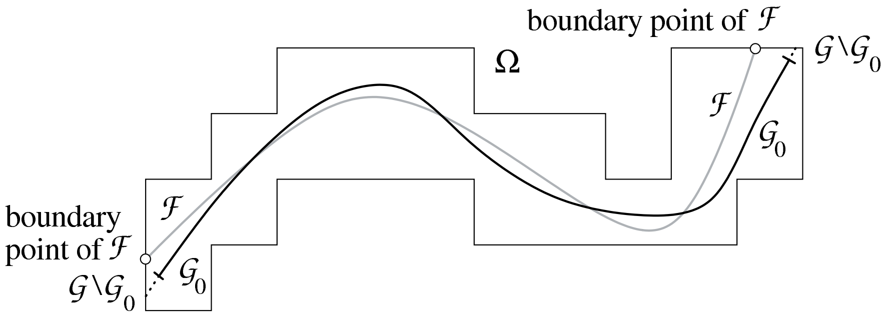

Fig. 2. The curve segments $\mathcal{F}$ (grey), $\mathcal{G}\backslash\mathcal{G}_0$ (dotted), $\mathcal{G}_0$ (solid), and the boundary points $\partial\mathcal{F}$ (marked by circles).

which is closely related to the Hausdorff distance (see [2])

$$\mathrm{Dist}^{\mathrm{H}}(\mathcal{F},\mathcal{G}) = \max\{\ \mathrm{dist}^{\mathrm{H}}(\mathcal{F},\mathcal{G}),\ \mathrm{dist}^{\mathrm{H}}(\mathcal{G},\mathcal{F})\ \}, \tag{15}$$

where $\mathrm{dist}^{\mathrm{H}}(\mathcal{F},\mathcal{G}) = \sup_{\mathbf{y}\in\mathcal{G}} \inf_{\mathbf{x}\in\mathcal{F}} \mathrm{dist}(\mathbf{x},\mathbf{y})$, as follows. The left part of the Hausdorff distance can be bounded by

$$\mathrm{dist}^{\mathrm{H}}(\mathcal{F},\mathcal{G}) \leq \max\{\mathrm{dist}^{\mathrm{F}}(\mathcal{F},\mathcal{G}), \underbrace{\sup_{\mathbf{x}\in\mathcal{G}\backslash\mathcal{G}_0} \inf_{\mathbf{y}\in\partial\mathcal{F}} \mathrm{dist}(\mathbf{x},\mathbf{y})\}}_{(*)}, \tag{16}$$

where $\partial\mathcal{F}$ is the set of all (finitely many) boundary points of $\mathcal{F}$, cf. Fig. 2. If the minimum distance of a point $\mathbf{y}\in\mathcal{G}_0$ from $\mathcal{F}$ always occurs at one of its footpoints (and not at the boundary points $\partial\mathcal{F}$), then the inequality (16) becomes an equation. This is a realistic assumption in applications. Consequently, if one ignores the contributions $(*)$ at the boundary, the Hausdorff distance of the curves $\mathcal{F}$ and $\mathcal{G}$ is essentially equal to the footpoint distance (14).

Theorem 3. *Let $K = \max_{(i,j)\in\mathcal{I}} |p_{i,j} - q_{i,j}|$, and assume (7), (8). The domain Ω is assumed to contain the line segments connecting the points in $\mathcal{G}_0$ with their footpoints on $\mathcal{F}$. Let h be chosen such that $h \geq C/D_h\, K$. The one–sided footpoint distance is then bounded by*

$$\mathrm{dist}^{\mathrm{F}}(\mathcal{F},\mathcal{G}) \leq \frac{C}{D_h}\, K. \tag{17}$$

Proof: Consider a point $\mathbf{y}\in\mathcal{G}_0$, hence $g(\mathbf{y}) = 0$. Using the convex hull property of B-splines we obtain $|f(\mathbf{y})| = |f(\mathbf{y}) - g(\mathbf{y})| \leq K$. Inequality (17) now follows from Corollary 2. □

If the curves $\mathcal{F}$ and $\mathcal{G}$ are sufficiently close to each other, then the difference $f(\mathbf{x}) - g(\mathbf{x})$ can be expected to be small, provided that the sign distributions of f and g are similar. Consequently, Theorem 3 can be expected to give tight upper bound for the distance of the curves.

Using this theorem, we are now able to derive bounds on $\mathrm{Dist}^{\mathrm{F}}(\mathcal{F},\mathcal{G})$, directly from the control points of the bivariate spline functions f and g. An example is given in Section 7.

§5. Distance of Implicitly Defined and Parametric Curves

Once again, let $\mathcal{F}$ be a planar curve defined by (1) and (2), and let $\mathcal{H}$ be a parametric B-spline curve of degree k (see [7]),

$$\mathcal{H} = \{ \mathbf{h}(t) \mid t \in [0,1] \}, \tag{18}$$

with the parametric representation $\mathbf{h}(t)$ and parameter domain $[0,1]$. Consider the spline function of degree $2dk$ which is obtained by restricting $f(\mathbf{x})$ to the curve $\mathbf{h}(t)$,

$$f(\mathbf{h}(t)) = \sum_{i=0}^{m} P_i^{2dk}(t)\, h_i, \quad t \in [0,1], \tag{19}$$

where the B-splines $P_i^{2dk}(t)$ of degree $2dk$ are defined over an appropriate knot sequence. In addition to the original knots of $\mathbf{h}(t)$, it contains the parameter values of the intersections of $\mathcal{H}$ with the knot lines of the bivariate spline function $f(\mathbf{x})$, both with sufficient multiplicity. For instance, if $\mathbf{h}(t)$ is a cubic spline curve, then the parameter values of the intersections with the knot lines can be found by solving a couple of cubic equations.

The coefficients $h_i \in \mathbb{R}$ can be computed with the help of algorithms for composing spline functions. A blossoming–based approach has been described in [3]. Alternatively, one may construct the B-spline representation of $f(\mathbf{h}(t))$ with the help of interpolation techniques.

The one–sided footpoint distance $\operatorname{dist}^{\mathrm{F}}(\mathcal{F}, \mathcal{H})$ can be defined as in (13). It is a useful distance measure in several applications, such as the approximate implicitization of the parametric B-spline curves $\mathcal{H}$, see [4], or for the approximate parameterization of the implicit curve $\mathcal{F}$. With the help of the control points h_i we obtain the following upper bound.

Theorem 4. *Let $H = \max_{i=0,\dots,m} |h_i|$, and assume (7), (8). The domain Ω is assumed to contain the line segments connecting the points in $\mathcal{H}_0$ (which is defined as $\mathcal{G}_0$) with their footpoints on $\mathcal{F}$. Let h be chosen such that $h \geq C/D_h\, K$. The distance of the points of $\mathcal{H}_0$ from their footpoints is then bounded by*

$$\operatorname{dist}^{\mathrm{F}}(\mathcal{F}, \mathcal{H}) \leq \frac{C}{D_h}\, H. \tag{20}$$

Proof: Consider a point $\mathbf{y} \in \mathcal{H}_0$, hence $\mathbf{y} = \mathbf{h}(t_0)$ for some $t_0 \in [0,1]$. Using the convex hull property of B-splines we obtain $|f(\mathbf{y})| = |f(\mathbf{h}(t_0))| \leq H$. Inequality (20) now follows from Corollary 2. □

§6. Generating the Bounds C and D_h

In order to generate the constants C and D_h in the inequalities (7) and (8), we split the spline function (1) into polynomial segments with the subdomains $\Omega^{(k)}$,

$$f(\mathbf{x}) = f^{(k)}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega^{(k)}, \quad k = 1, \dots, K. \tag{21}$$

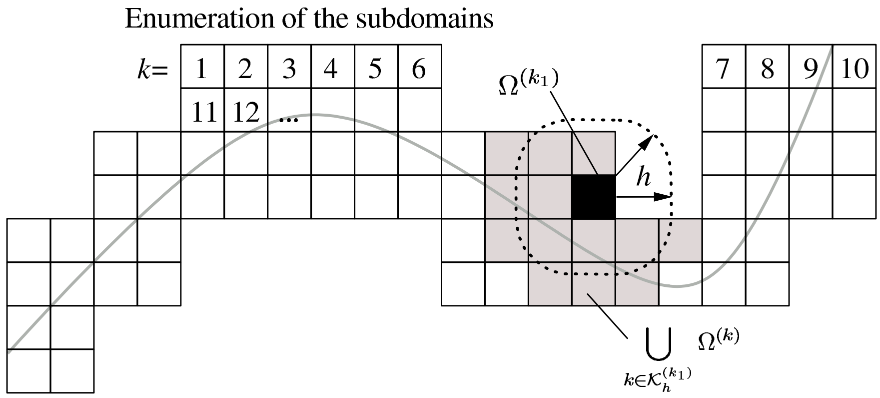

Fig. 3. Splitting the tensor–product spline function into polynomial (tensor–product Bézier) segments, and definition of the index sets $\mathcal{K}_h^{(k)}$.

The K subdomains $\Omega^{(k)}$ of the polynomial pieces are either the cells of the original spline function, or they are obtained by splitting them further into smaller rectangles. Using smaller subdomains $\Omega^{(k)}$, instead of the cells of the spline function, we may obtain tighter bounds C and D_h.

As an example, Fig. 3 shows the enumeration of the subdomains $\Omega^{(k)}$ which are used for splitting the spline function of Fig. 1a into polynomial pieces. Here, they are obtained by subdividing each of the original quadratic cells into four smaller squares.

Using knot insertion we obtain for each subdomain a tensor–product Bézier representation

$$f^{(k)}(\mathbf{x}) = \sum_{i=0}^{d} \sum_{j=0}^{d} B_i^d(x_1)\, B_j^d(x_2)\, b_{i,j}^{(k)}, \quad \mathbf{x} \in \Omega^{(k)}, \tag{22}$$

with certain coefficients $b_{i,j}^{(k)}$. Moreover, we may compute a tensor–product Bézier representation of the associated gradients

$$\nabla f^{(k)}(\mathbf{x}) = \sum_{i=0}^{d} \sum_{j=0}^{d} B_i^d(x_1)\, B_j^d(x_2)\, \mathbf{c}_{i,j}^{(k)} \quad \mathbf{x} \in \Omega^{(k)}, \tag{23}$$

where the control points $\mathbf{c}_{i,j}^{(k)}$ are obtained from the formulas for differentiation and degree elevation of tensor–product polynomials in Bernstein–Bézier form. Note that the first (resp. second) component is a polynomial of degree $(d-1, d)$ (resp. $(d, d-1)$). Thus, degree elevation is needed in order to obtain the representation of the form (23).

Inequality (8) involves gradients at two points with a certain maximum distance h. For each subdomain $\Omega^{(k)}$, we denote with $\mathcal{K}_h^{(k)}$ the indices of all subdomains that are within distance h of it,

$$i \in \mathcal{K}_h^{(k)} \iff \exists\, \mathbf{x} \in \Omega^{(k)}\ \exists\, \mathbf{y} \in \Omega^{(i)} : \ \mathrm{dist}(\mathbf{x}, \mathbf{y}) \le h. \tag{24}$$

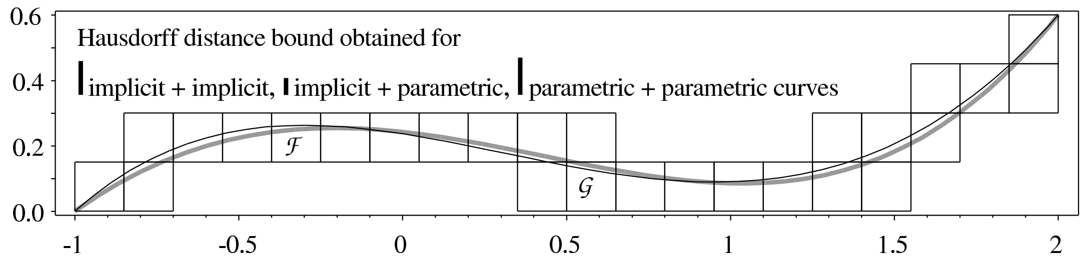

Fig. 4. The Hausdorff distance bounds which are obtained for two cubic Bézier curves by representing them in implicit and/or parametric form.

Geometrically, the set $\mathcal{K}_h^{(k)}$ contains the indices of all subdomains which have points within the offset curve (which consists of line segments and circular arcs) at distance h of the boundary of $\Omega^{(k)}$, see Fig. 3.

Lemma 5. *The inequalities (7) and (8) are valid with the following constants:*

$$C = \max_{\substack{k=1,\dots,K\\ i,j=0,\dots,d}} \|\mathbf{c}_{i,j}^{(k)}\| \quad \textit{and} \quad D_h = \min_{\substack{k_1=1,\dots,K;\ k_2\in\mathcal{K}_h^{(k_1)}\\ i_1,i_2,j_1,j_2=0,\dots,d}} \mathbf{c}_{i_1,j_1}^{(k_1)} \cdot \mathbf{c}_{i_2,j_2}^{(k_2)}. \tag{25}$$

Proof: These constants are obtained by applying the convex hull property of polynomials in Bézier form to the gradient (23) and to the inner product of gradients

$$\nabla f(\mathbf{x}_1)\cdot\nabla f(\mathbf{x}_2), \quad \mathbf{x}_1\in\Omega,\ \mathbf{x}_2\in\Omega,\ \mathrm{dist}(\mathbf{x}_1,\mathbf{x}_2)\le h. \quad \square \tag{26}$$

§7. Examples and Conclusions

We apply the theoretical results to the two cubic Bézier curves $\mathcal{F}$ and $\mathcal{G}$ which are shown in Fig. 4. Both curves have identical segment–end points and are fairly close together. Thus we have $\mathcal{F}=\mathcal{F}_0$, $\mathcal{G}=\mathcal{G}_0$, and the minimum distance of a point on one curve to the other curve always occurs at its footpoint. Consequently, the Hausdorff distance (15) is equal to both one–sided footpoint distances $\mathrm{dist}^{\mathrm{F}}(\mathcal{F},\mathcal{G})$ and $\mathrm{dist}^{\mathrm{F}}(\mathcal{G},\mathcal{F})$.

Originally, the curves are given as cubic Bézier curves $\mathbf{f}(t)$ and $\mathbf{g}(t)$, both with the domain [0,1]. After implicitizing them, we obtain (bi-) cubic tensor–product polynomials $f(\mathbf{x})$ and $g(\mathbf{y})$. In order to obtain suitable constants C and D_h, we choose the domain Ω as the union of all squares (size 0.15×0.15) which are shown in the figure. Within each square, the functions f and g are represented in tensor–product Bézier form.

As outlined in Section 6, we generate the constants C and D_h for both curves, where $h=0.15$. This eventually gives the following bounds on the Hausdorff distance of both curves:

(a) Theorem 3 (pair of implicitly defined curves): $\mathrm{Dist}^{\mathrm{H}}(\mathcal{F},\mathcal{G})\le 0.108$.

(b) Theorem 4 (implicit and parametric curve): $\mathrm{Dist}^{\mathrm{H}}(\mathcal{F},\mathcal{G})\le 0.052$. Here, the composition $f(\mathbf{g}(t))$ yields a polynomial of degree 9. The upper bound H is found as the maximum absolute value of the control points which are obtained after splitting it uniformly into four Bézier segments.

Both bounds should be compared with the one which results directly from the parametric representations:

(c) From the convex–hull property (pair of parametric curves):

$$\mathrm{Dist}^{\mathrm{H}}(\mathcal{F},\mathcal{G}) \leq \max_{t\in[0,1]} \|\mathbf{f}(t) - \mathbf{g}(t)\| \leq 0.119. \tag{27}$$

The upper bound is obtained as the maximum length of the difference vectors of the corresponding control points which are obtained after splitting both curves uniformly into four Bézier segments.

The three bounds are shown in Fig. 4. The second bound is fairly close to the exact Hausdorff distance of both curves.

The above bounds depend on the choice of the constant h, which is an initial estimate of the Hausdorff distance. If a parametric representation of both curves is available, then the corresponding bound (c) can serve as an initial value for h. Alternatively one may use discretization–based or heuristical techniques. This should then be combined with an iterative adaptation technique.

The bound (b), obtained by combining implicit and parametric representations is the tightest one. Theorem 4 will be useful to obtain error bounds for the approximate implicitization and approximate parameterization of algebraic spline curves and surfaces, cf. [1,4].

The bounds (a) and (c), obtained by using either the implicit representations or the parametric ones, are almost identical. However, it should be noted that Theorem 3 provides distance bounds for the more general class of algebraic spline curves, whereas the parameterization–based techniques can deal only with rational ones. Also, the parameterizations of the two curves are relatively similar, as they have identical segment end points. In the general situation the parameterization–based approach is expected to give less accurate results.

The techniques presented in this paper can be used only if, at least within a neighbourhood of the curve, the gradients $\nabla f(\mathbf{x})$, $\nabla g(\mathbf{x})$ of the functions $f(\mathbf{x})$, $g(\mathbf{x})$ satisfy certain regularity conditions. Ideally, these gradients would all be unit vectors. Then, the function $f(\mathbf{x})$ would be simply the signed distance function, or 'normal form', of the curve. See [5,6] for further information on normal forms and their applications. Theorems 3 and 4 can be applied to functions whose gradient field is not too different from the ideal situation. In particular, points with vanishing gradients (extrema and saddle points) of the functions are $f(\mathbf{x})$, $g(\mathbf{x})$ have to be excluded.

In order to obtain tighter bounds, the gradient fields could be improved by multiplying the defining functions with suitable bivariate polynomials.

The curve fitting algorithms described in [8] and [11] produce bivariate spline functions whose gradients approximate unit vectors along its zero contour (i.e., along the corresponding algebraic spline curve). In addition to avoiding singularities, this makes these functions well suited for applying Theorems 3 and 4.

As a matter of future research, we plan to extend the results of this paper to algebraic spline surfaces.

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