Construction and modification of convex parametric spline curves and surfaces

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Abstract: We give an outline of several methods for the construction and modification of parametric spline curves and surfaces. These methods are based on the use of linearized convexity conditions. With the help of a so–called reference curve or surface it is possible to find data–dependent systems of linear inequalities for the control points which imply the desired convexity properties. As a consequence, several problems of shape–preserving curve or surface construction and modification can be formulated as optimization problems with linear constraints.

1 Introduction

Variational Design of curves and surfaces has become one of the standard techniques of Computer Aided Geometric Design. One of its origins is the conference article [5] article by Hagen and Schulze. In this volume, the paper by Greiner gives a survey of the various methods of variational design.

As the basic idea of variational design, a curve or surface is constructed by minimizing a suitable energy (or fairness) functional subject to certain side–conditions, e.g. interpolation or approximation of given data. Ideally one would like to choose highly non–linear functionals like the elastic bending energy

$$\mathcal{E}_2 = \int_{s_0}^{s_1} \kappa^2 \, \mathrm{d}s \tag{1}$$

of a planar parametric curve. The use of such exact non–linear functionals, however, leads to optimization problems which are relatively difficult to deal with. For this reason, many methods rely on simpler functionals like the linearized bending energy

$$E_2 = \int_{t_0}^{t_1} \left\| \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} \right\|^2 \, \mathrm{d}t. \tag{2}$$

These functionals lead to quadratic functions of the spline coefficients / control points. For instance, if E_2 is to be minimized subject to interpolation conditions, then the optimal curve or surface can easily be found by solving a system of linear equations.

An interesting method has been proposed by Greiner [4]. With the help of a *reference curve* (or surface) he constructs a quadratic data–dependent functional which approximates the exact thin–plate energy of a surface. Based on a similar data–dependent linearization technique, Kobbelt [13] has developed an efficient interpolatory subdivision scheme which produces fair surfaces. As the major advantage of these data–dependent quadratic functionals, the approximate minimization of the non–linear fairness measures becomes accessible for numerical computations, even if the number of data comes up to the order which is needed in applications.

The methods of variational design are particularly well suited for generating *fair* curves and surfaces. One the other hand, in many applications there is an increasing need for creating curves and surfaces which are subject to shape constraints like convexity. For instance, in reverse engineering a CAD model of an object is to be (re–) constructed from a cloud of measurement data [6, 16]. A typical strategy consists of two steps. After a rough segmentation of the data according to certain geometric features, several surface patches are fitted to the individual segments. In order to improve the result of the second step it can be crucial to use additional knowledge on the convexity of the surface patches. See Figure 2 of [8] for an illustration.

In the case of parametric curves or surfaces, the exact convexity constraints are highly non–linear and they produce optimization problems which are rather difficult to deal with. In the sequel we give an outline of linearization techniques for various shape constraints and their application. We discuss convexity of planar parametric curves and of parametric surface patches.

Note that the linearization technique has some common features with the above–mentioned work by Greiner and Kobbelt [4, 13] on data–dependent "fairness" or "energy" functionals. In the sequel we discuss data–dependent linear approximations of non–linear shape constraints. Whereas the details of the linearization techniques have been described elsewhere, in this paper we focus on the general idea and we give some additional geometric interpretations.

2 Curve fitting with convex parametric spline curves

A method for shape–preserving least–squares approximation of planar data with planar polynomial spline curves has been developed in [9], see also [8] for an outline of the scheme. It produces a parametric B–spline curve of degree d,

$$\mathbf{x}(t) = \sum_{i=0}^{m} N_i^d(t)\ \mathbf{d}_i, \quad t \in [t_0, t_1] \subset \mathbb{R}, \tag{3}$$

with the control points $\mathbf{d}_i \in \mathbb{R}^2$ and the B–spline basis functions $N_i^d(t)$ (see [7]), which approximates given planar data $(\mathbf{p}_j)_{j=0,\dots,n}$ with associated parameters $\tau_j \in [t_0, t_1]$. These parameters can be estimated from the data [7]. The approximation is to satisfy shape constraints of the type

$$\det\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{x}(t), \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathbf{x}(t)\right) \geq 0 \quad \text{or} \quad \leq 0 \quad \text{for certain intervals} \quad t \in [t_a, t_b] \subseteq [t_0, t_1]. \tag{4}$$

That is, for certain segments one can specify the curvature sign of the curve.

In the sequel we give an additional geometric interpretation of the method. The expected shape of the approximating curve is described by the so–called *reference curve*. This curve specifies both the desired inflections of the curve and the expected curvature signs. It is used for generating a system of linear inequalities which guarantee the desired shape. The construction is based on bounding wedges for the first and second derivative vectors, see [9] for details. Note that the linearized shape constraints *depend* on the *data*, because

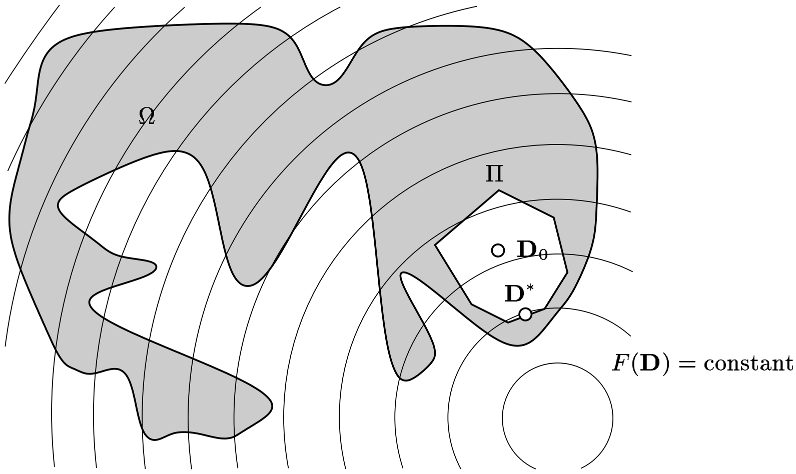


Fig. 1 A schematic illustration of the shape–preserving curve fitting scheme.

the reference curve is data–dependent. Thus, our construction is an analogue to Greiner's construction of data–dependent energy functionals.

Figure 1 gives a schematic illustration of the method. The planar parametric spline curves which have the specified inflections and curvature signs form a certain subset Ω of the real linear space $\mathbb{R}^{2(m+1)}$, where the coordinates of the points $\mathbf{D}$ are obtained by collecting the B–spline coefficients,

$$\mathbf{D} = (\ \mathbf{d}_0\ \mathbf{d}_1\ \dots\ \mathbf{d}_m\ ). \tag{5}$$

The reference curve corresponds to a point $\mathbf{D}_0$ of this space. The linearized shape constraints describe a polyhedron $\Pi \subset \Omega$ which is circumscribed to the reference curve $\mathbf{D}_0$. As shown in [9], under certain assumptions made about the reference curve, it is always possible to construct such a circumscribed polyhedron.

With the help of the linearized convexity constraints, the task of shape–preserving curve fitting can be formulated as a quadratic programming problem: a quadratic objective function is to be minimized subject to linear constraints. The objective function can be chosen as a combination of the least squares sum with a weighted smoothing term, e.g.,

$$F(\mathbf{D}) = \sum_{j=0}^{n} \|\mathbf{x}(\tau_j) - \mathbf{p}_j\|^2 + w \int_{t_0}^{t_1} \|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x}(t)\|^2\,\mathrm{d}t \tag{6}$$

with the weight $w > 0$. There exists a number of fast and efficient algorithms for solving optimization problems of this type, see [2, 15]. A schematic illustration is again given in Figure 1 where the objective function $F(\mathbf{D})$ is visualized by its level curves.

After a first solution has been computed, the whole procedure can be iterated in order to obtain better results. By using the first results as a new reference curve one obtains linearized shape constraints which are better adapted to the data. Simultaneously one may insert knots (that is, add degrees of freedom) in order to improve the approximation result. Typically after a few iterations one gets the desired result.

A quadratic–programming based approach to shape–preserving curve fitting with spline *functions* has been developed by Dierckx [1]. Our method can be seen as a generalization of Dierckx' scheme to the parametric setting.

3 Linearized convexity conditions for parametric surfaces

As outlined in the preceding section, linear sufficient convexity conditions for polynomial parametric spline curves can be derived with the help of bounding wedges for the first and second derivative vectors of the curve. These bounding wedges guarantee, that the second derivative vectors always "point to the same side of the tangent". This implies local convexity of the curve. If the tangent of the curve does not vary too much, then local convexity implies global convexity.

This idea can be applied to parametric surface patches. For example, consider a parametric tensor–product Bézier surface patch of degree (m, n)

$$\mathbf{x}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u)\, B_j^n(v)\, \mathbf{b}_{i,j}, \qquad (u,v) \in [0,1]^2, \tag{7}$$

with the control points $\mathbf{b}_{i,j} \in \mathbb{R}^3$, see [7]. The partial derivative vectors of this surface patch are governed by certain difference vectors of the control points. Unlike the curve case, now one has to deal with a *system* of second directional derivative vectors,

$$\begin{aligned} \mathbf{x}''(u_0, v_0, \xi, \eta) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x}(\, u_0 + t\,\xi, v_0 + t\,\eta\,)\Big|_{t=0} \\ &= \xi^2\, \mathbf{x}_{uu}(u_0,v_0) + 2\,\xi\,\eta\, \mathbf{x}_{uv}(u_0,v_0) + \eta^2\, \mathbf{x}_{vv}(u_0,v_0), \end{aligned} \tag{8}$$

with respect to all directions $(\xi, \eta) \in \mathbb{R}^2 \setminus \{(0,0)\}$. If all these vectors "point to the same side of the tangent plane", that is, if either

$$\mathbf{x}''(u,v,\xi,\eta) \cdot (\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)) \geq 0 \quad \text{or} \quad \leq 0 \tag{9}$$

holds for all $(u,v) \in [0,1]^2$, $(\xi,\eta) \in \mathbb{R}^2$, then the Bézier surface patch is guaranteed to be locally convex. Note that multiplying the right–hand side of (9) with the normalizing factor $1/\|\mathbf{x}_u \times \mathbf{x}_v\|$ would produce nothing but the second fundamental form of the surface patch, see [7] or any textbook on differential geometry. Thus, the inequalities (9) guarantee that the surface has either non–negative or non–positive normal curvatures.

In order to find a sufficient system of linear inequalities we proceed as follows:

• **Step 1.** Choose *bounding polyhedral cones* for the control nets of the first derivative vectors $\mathbf{x}_u$ and $\mathbf{x}_v$, see Figure 2 for an illustration. This leads to linear inequalities for the components of the Bézier control points $\mathbf{b}_{i,j}$. Simultaneously, under suitable assumptions about the bounding cones of the first derivatives, one gets another bounding polyhedral cone for the normal vector $\mathbf{x}_u \times \mathbf{x}_v$. That is, the normal vector is guaranteed to be a non–negative linear combination of certain bounding vectors $(\vec{\mathbf{r}}_k)_{k=0,\dots,p}$,

$$\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v) = \sum_{k=0}^{p} \lambda_k(u,v)\, \vec{\mathbf{r}}_k \quad \text{with} \quad \lambda_k(u,v) \geq 0. \tag{10}$$

The bounding vectors $(\vec{\mathbf{r}}_k)_{k=0,\dots,p}$ define the *feasible region* for the second directional derivative vectors, see Figure 2. If the inequalities

$$\mathbf{x}''(u,v,\xi,\eta) \cdot \vec{\mathbf{r}}_k \geq 0 \quad \text{or} \quad \leq 0, \quad k = 0, \dots, p, \tag{11}$$

are satisfied, then the Bézier surface patch is guaranteed to be locally convex.

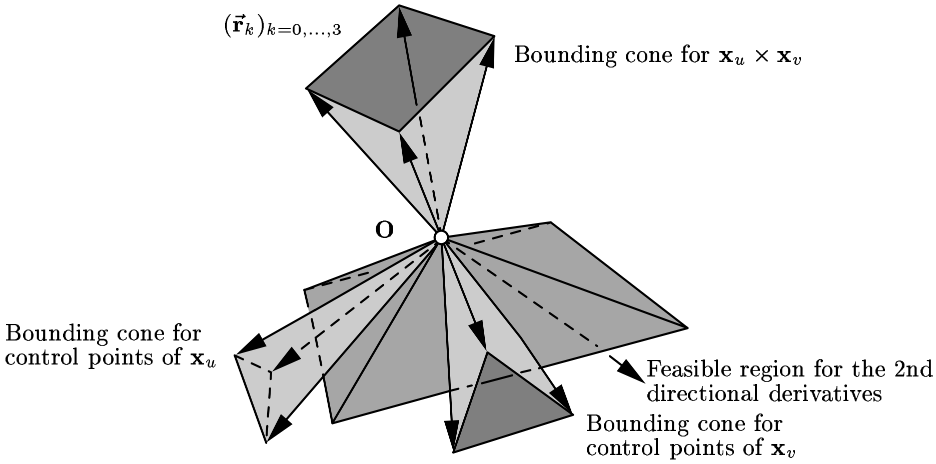


Fig. 2 Linearized convexity conditions for parametric surfaces.

• **Step 2.** Choose a *parameterization* of the directions (ξ, η) of the second directional derivatives. In fact, it suffices to guarantee the inequalities (11) for a subset $D \subset \mathbb{R}^2$ provided that this subset spans the plane $\mathbb{R}^2$. That is, we guarantee (11) for all $(\xi, \eta) \in D$, where $\mathbb{R}D = \mathbb{R}^2$.

One may simply choose D as the union $D = \bigcup_{r=1}^{q} D_r$ of suitable line segments

$$D_r = \{ \, (\, (1-t)\, \xi_{r-1} + t\, \xi_r, \; (1-t)\, \eta_{r-1} + t\, \eta_r \,) \mid t \in [0,1] \, \}, \quad r = 1, \ldots, q, \tag{12}$$

see Figure 3 for an illustration. For the r–th line segment we obtain from (8) the second directional derivatives

$$\mathbf{x}''(u, v, \ldots) = (1-t)^2 \, \mathbf{c}_{r,0}(u,v) + 2\, t\, (1-t)\, \mathbf{c}_{r,1}(u,v) + t^2 \, \mathbf{c}_{r,2}(u,v) \tag{13}$$

with

$$\begin{aligned} \mathbf{c}_{r,0} &= \xi_{r-1}^2 \, \mathbf{x}_{uu} + 2\, \xi_{r-1}\, \eta_{r-1} \, \mathbf{x}_{uv} + \eta_{r-1}^2 \, \mathbf{x}_{vv}, \\ \mathbf{c}_{r,1} &= \xi_{r-1}\, \xi_r \, \mathbf{x}_{uu} + (\xi_{r-1}\, \eta_r + \xi_r\, \eta_{r-1}) \, \mathbf{x}_{uv} + \eta_{r-1}\, \eta_r \, \mathbf{x}_{vv} \quad \text{and} \\ \mathbf{c}_{r,2} &= \xi_r^2 \, \mathbf{x}_{uu} + 2\, \xi_r\, \eta_r \, \mathbf{x}_{uv} + \eta_r^2 \, \mathbf{x}_{vv}. \end{aligned} \tag{14}$$

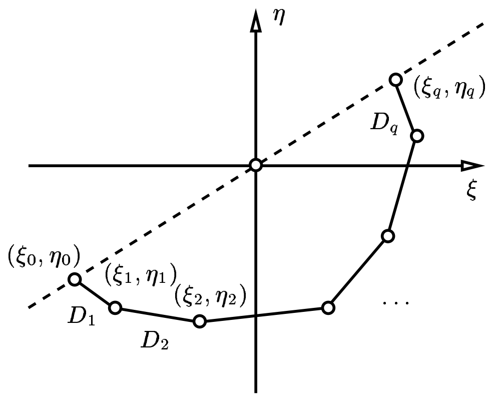


Fig. 3 Parameterization of the directions (ξ, η) of the second directional derivatives.

These coefficients are vector–valued polynomials of degree $\leq (m, n)$. They have a tensor–product representation (cf. (7)) with certain control points $(\mathbf{d}_{r,l,i,j})_{i=0,\ldots,m;j=0,\ldots,n}$ $(r = 1, \ldots, q,\ l = 0, \ldots, 2)$. These control points are certain linear combinations of the Bézier control points $\mathbf{b}_{i,j}$.
If the control points $\mathbf{d}_{r,l,i,j}$ satisfy the linear inequalities

$$\mathbf{d}_{r,l,i,j} \cdot \vec{\mathbf{r}}_k \geq 0 \quad \text{or} \quad \leq 0 \quad (k = 0, \ldots, p) \tag{15}$$

then the second directional derivative vectors are contained within the feasible region, see Figure 2. Hence, under this assumption, the Bézier surface patch is guaranteed to be locally convex.

These two steps lead to a system of linear sufficient convexity conditions for parametric Bézier surfaces. In our implementation, both the bounding polyhedral cones for the first derivative vectors and the parametrization of the directions (ξ, η) of the second directional derivatives are chosen with the help of a strongly convex *reference surface*, see [11] for details. As in the curve case we get a system of *data–dependent* linear inequalities for the Bézier coefficients $\mathbf{b}_{i,j}$.
As observed in [11], the linearized convexity conditions can be adapted to any strongly convex Bézier surface patch. For this adaptation, however, one may need to subdivide the surface into several segments.
Once again, Figure 1 may serve as a schematic illustration of the method. The convex parametric Bézier surfaces form a certain subset Ω of the real linear space $\mathbb{R}^{3(m+1)(n+1)}$, where the coordinates of the points are obtained by collecting the Bézier control points. The strongly convex surfaces correspond to a point $\mathbf{D}_0$ of this space. The linearized convexity constraints describe a polyhedron $\Pi \subset \Omega$ which is circumscribed to $\mathbf{D}_0$. From this point of view, the existence of linear sufficient convexity conditions is rather obvious; each inner point of Ω has a circumscribed polyhedron. The main difficulty, however, is caused by the relatively complicated structure of the set Ω. It is shown in [11] that the linearization procedure is compatible with the structure of Ω.
Based on the above linearization procedure, a method for fitting convex parametric Bézier surfaces to scattered data has been developed in [12]. The control points of the Bézier patch are found by minimizing the least–squares sum subject to the linearized convexity conditions. This method generalizes the curve fitting algorithm from [9] to the case of parametric surfaces. In the initial step of the method, one has to find a strongly convex reference surface. The reference surface is constructed by minimizing another quadratic functional without constraints. This quadratic functional is a weighted linear combination of a suitable tension term and the least–squares sum.
In many cases of practical interest, convex shapes can be described by spline *functions* rather than parametric surfaces. The above linearization procedure can be used in order to find linearized convexity conditions for piecewise tensor–product polynomials too. In this (much simpler) case, the first step is omitted; the second step can be applied to the second directional derivatives. The details of the linearized convexity conditions, and their application to shape–preserving surface fitting are discussed in the forthcoming article [10]. Unlike the parametric case, the convex piecewise polynomial tensor–product functions form a convex cone, as a positive linear combination of two convex functions is again guaranteed to be convex. As observed in [10], the linearized convexity conditions are able to approximate the set of convex tensor–product polynomials as good as desired.

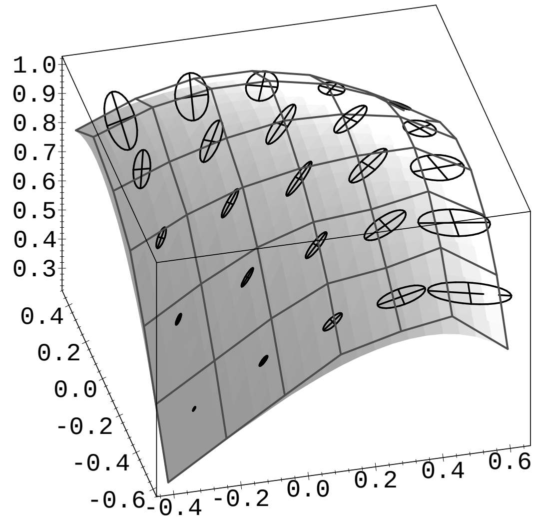
4 Convexity–preserving modification of convex surfaces

In this section we describe an application of the linear sufficient convexity conditions. A convex (or at least partly convex) parametric Bézier or B–spline surface patch is assumed to be given. During the process of surface design, the user may wish to modify this surface patch. For instance, he may pick a number of points on the surface and specify new positions for them. In addition, certain boundary conditions (e.g. boundary curves and cross–derivatives) are given. Moreover, the modification is to preserve the convexity of the surface.

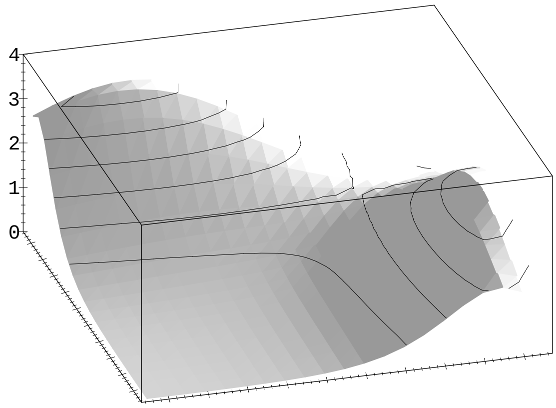
This type of surface modification has been discussed by Schichtel in his Ph.D. thesis [14]. He dubbed this process the convexity–preserving "lifting" of a surface patch. With the help of the above linearization procedure we develop an optimization–based approach to convexity–preserving surface modification which in many cases works better than Schichtel's methods. Firstly we use the initial surface as a reference surface and we generate a data–dependent system of linear sufficient convexity conditions. An outline of this procedure has been given in the preceding section. Secondly we choose a suitable quadratic or linear objective function which reflects the intended surface modification. For instance, one may use the sum of the squared distances between the initial surface points and their desired conditions. Now, the control points of the modified surface patch result from linear or quadratic programming, as outlined in Section 2. Once again, Figure 1 provides a schematic illustration of the lifting scheme.

As an example, consider the convex Bézier surface patch of degree (6,6) which is shown in Figure 4e. The ellipses in the figure visualize the distribution of the normal curvatures. Whereas the principal diameters are the principal curvature directions, the length of the diameters is chosen proportional to the principal curvatures. As a consequence, the area of the ellipses represents the Gaussian curvature. The initial surface was used in order to generate linear sufficient convexity conditions. The procedure from the preceding section led to a system of 1196 inequalities. As we imposed C^1 boundary conditions along the two boundaries in the background, only 75 of components of control points were free for the optimization. The Figures 4a and c show the surface patches which are obtained after the left corner has been pulled downwards and upwards, both times subject to the linearized convexity conditions. The Gaussian curvatures of the modified surface patches have been plotted in Figures 4b and d. For both modifications, the optimization using LOQO [15] took approximately 5 seconds of CPU time on a HP 715/64 workstation. This example illustrates the flexibility of the linearized convexity constraints. In between the two extreme shapes, the linearized convexity conditions offer a broad spectrum of feasible convex surfaces.

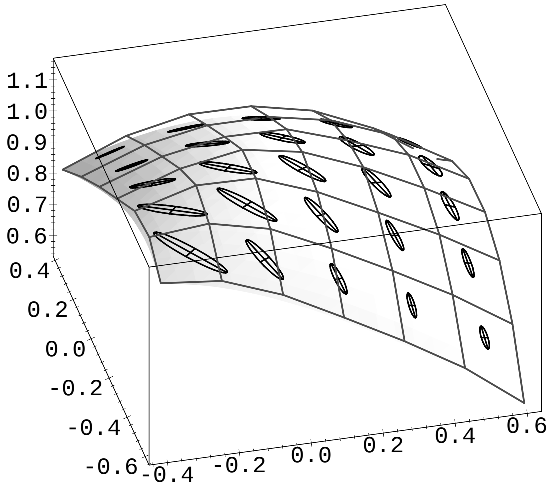
In our numerical experiments, the above algorithm for convexity–preserving surface modification led to very good results provided that the initial surface was relatively far away from having parabolic points. In the latter cases, however, the linearization procedure can produce a large number of inequalities, as a large number of subdivisions (both of the directions in the parameter domain and of the surface patch) may be required. In addition, the linearized constraints can then be fairly restrictive, and the set of feasible surfaces may get too small for the intended modification. It these cases the use of non–linear shape constraints might be more appropriate. Such constraints are described in the contribution by Kaklis and Koras to this volume. Of course one has to use much more sophisticated methods for the numerical optimization.



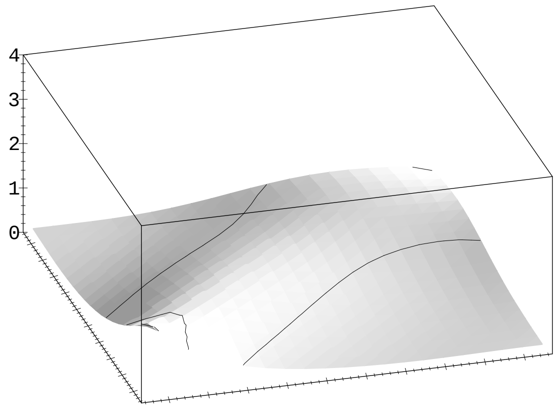
(a)



(b)



(c)



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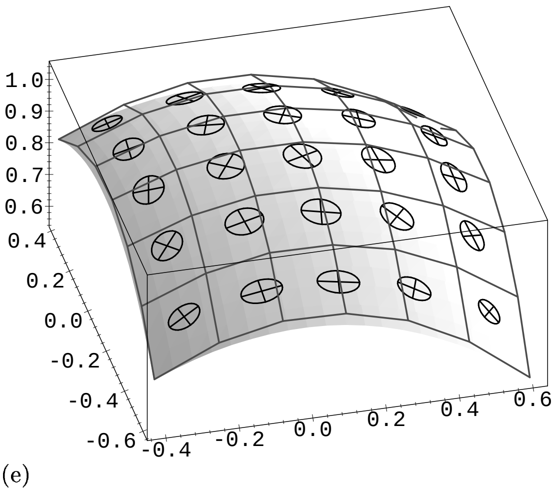
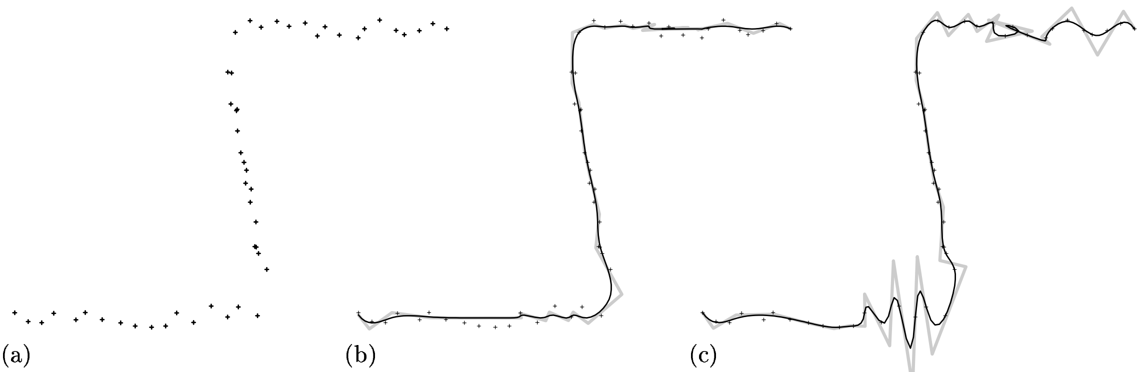


(e)

Fig. 4 Convexity preserving surface modification. The initial surface (e), after lifting it downwards (a) and upwards (c) and the Gaussian curvature distribution (b,d) of the modified surfaces. The ellipses visualize the distribution of the normal curvature.

5 Concluding remarks

In this article we have outlined techniques for generating linear sufficient convexity conditions, both for planar parametric spline curves and for parametric Bézier and B–spline surface patch. In addition, we described their application to the shape–preserving construction and modification of parametric curves and surfaces.
As a common feature of the linearization techniques, they lead to a data–dependent system of linear inequalities which guarantee the desired shape properties. That is, we are able to replace the non–linear exact shape constraints with suitable linear inequalities. Then, in many examples, an approximate solution to the problem can be found in a reasonable amount of computing time. Of course, the quality of the result heavily depends on the reference curve or surface which is used in the linearization procedure.
The above linearization techniques can be applied to other types of shape constraints also. Spline surfaces with convex level curves $z =$ constant have been studied in [8]. Based on the linear sufficient convexity conditions one can formulate methods for interpolation and approximation of given data by such surfaces.
Finally, one may easily derive linearization techniques for so–called *range constraints*, see [3] or the article by Schmidt in this volume. For a subset of the parameter domain, the curve (or surface) is requested to lie on one side of a given line (or plane). This can easily be guaranteed by introducing feasible regions for the control points of a parametric curve or surface. In addition, by using artificial subdivisions (that is, we insert new knots of the B–spline representation, without introducing new degrees of freedom) we are able to find weaker constraints.



(a) (b) (c)

Fig. 5 Least–squares approximation of planar data (a) with (b) and without (c) range constraints.

An example is shown in Figure 5. The data (a) are approximated by a planar parametric spline curve. The unconstrained approximation (c) has a lot of unwanted oscillations. The constrained approximation (b) has been computed after introducing range constraints. In its horizontal regions, the curve is restricted to a narrow rectangle along the data.

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