COMPUTATIONAL METHODS FOR DISCRETE PARAMETRIC $\ell_1$ AND $\ell_\infty$ CURVE FITTING

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The paper is devoted to $\ell_1$ and $\ell_\infty$ approximation with parametric spline curves. We discuss the questions of existence and uniqueness of solutions. With the help of a suitable linearization of the Euclidean norm, we derive a method for computing the approximating spline curves. The method uses linear and quadratic programming in order to find the solution.

Keywords: Curve fitting, parametric spline curves, $\ell_1$ – and $\ell_\infty$–approximation.

1. Introduction

The construction of curves (and surfaces) from scattered data is one of the fundamental tasks of Computer Aided Geometric Design. It has attracted a great deal of research, see [1, Section 4.4] for references. A typical strategy consists of two steps. Assume, that a sequence of points $(p_j)_{j=0,...,n}$ in the plane $\mathbb{R}^2$ is given. Firstly one associates a sequence of parameters $t_j$ with the data. Secondly, a suitable approximation scheme is applied to the coordinates of the data in order to find the approximating curve $x(t)$, e.g., a B–spline curve.

The majority of the spline fitting procedures described in the literature relies on the $\ell_2$–norm of the error vectors,

$$\sqrt{\sum_{j=0}^n |x(t_j) - p_j|^2},$$

where $\| \cdot \|$ is the Euclidean norm. (See Section 3 for more comments on the notation.) As an important advantage of this approach, the error function separates into independent terms for the two coordinates of the data. In addition, the solution can easily be computed by solving a system of linear equations. As a disadvantage, the geometric meaning of the objective function (1) is somewhat complicated. The contribution of a point to (1) increases quadratically with its distance from the
curve. That is, remote points (sometimes also called ‘outliers’; cf. Section 4.5) have a much bigger influence to the resulting curve than the closer ones.

In some applications it may be more appropriate to use other kinds of error functions, in order to obtain better shapes. For instance, as an obvious choice, one may wish to minimize the maximum distance of the points \( x(t_j) \) from the data. This will produce the curve which is as close as possible to the given data. However, as a well-known fact, the solution to this problem tends to oscillations. Another suitable objective function could be the overall length of the error vectors \( x(t_j) - p_j \). This is mainly advantageous for fitting so-called ‘uncertain data’ (as sometimes generated by optical scanning devices) which contain ‘outliers’, see [2]. These two possibilities lead to the discrete parametric \( \ell_1 \) (overall length) and \( \ell_\infty \) (maximum distance) approximation problems; they can be formulated with the help of the \( \ell_1 \) and the \( \ell_\infty \) norm of the sequence \( |x(t_j) - p_j| \).

In the case of approximation with spline functions, both tasks are well-studied problems in approximation theory. Section 2 provides an outline of these results. As one of the basic observations, the \( \ell_1 \) and \( \ell_\infty \) approximation of scattered data with spline functions can be formulated as linear programming problems, see the note by Barrodale and Young [3], cf. [4]. A more recent contribution to this subject is the conference article by Heidrich et al. [2].

The present paper is devoted to both theoretical and computational aspects of \( \ell_1 \) and \( \ell_\infty \) approximation with parametric spline curves. Firstly, in Section 2, we summarize some of the results on \( \ell_1, \ell_2, \ell_\infty \) data fitting with spline functions which are available in the literature. In Section 3 we formulate the approximation problems for parametric curves. We examine the questions of existence and uniqueness of solutions. It is shown that both problems do not have unique solutions in general. In order to guarantee uniqueness, we propose a modification of the original problems. The final section is devoted to computational aspects of the approximation problems. Based on a suitable linearization of the Euclidean norm we develop a method for parametric \( \ell_1 \) and \( \ell_\infty \) approximation. Finally, the scheme is illustrated by an example. We conclude the paper with a brief comparison of the different objective functions with respect to the shapes of the solutions.

2. Fitting Discrete Data with Spline Functions

A set of \( n+1 \) data \((t_j,p_j)\)\(_{j=0,...,n}\) with monotonically increasing abscissas, \( t_j < t_{j+1} \), is assumed to be given. We consider a spline function in B-spline representation,

\[
f(t) = \sum_{i=0}^{m} N_i^d(t) \ c_i, \quad t \in [t_0, t_n],
\]

of degree \( d \) with the \( m+1 \) B-spline coefficients \( c_i \in \mathbb{R} \), \( m \leq n \). The B-spline functions \( N_i^d(t) \) are defined with respect to a certain knot vector. For more information on spline functions and their B-spline representation the reader is referred to one of the various textbooks on this subject, e.g. [1, 5, 6]. The B-spline coefficients \( c_i \in \mathbb{R} \) are to be chosen so that the spline function approximates the given data.
Of course, firstly one has to generate suitable knots for the spline function. An adaptive algorithm is described in the textbook by Dierckx [7].

As the most popular choice, the spline coefficients are often computed by minimizing the $\ell_2$ norm of the error vector,

$$ (FA_2 : ) \quad \sum_{j=0}^{n} (f(t_j) - p_j)^2 = \| (f(t_j) - p_j)_{j=0,\ldots,n} \|_2 \to \text{Min} \tag{3} $$

This will be called the functional $\ell_2$ approximation problem $FA_2$. As an important advantage of this error function, the spline coefficients $c_i$ can be computed by solving a banded $(m+1) \times (m+1)$ system of linear equations. This system is formed by the so-called normal equations of the error function, see [8]. A unique solution can be shown to exist if and only if a subset of the abscissas $t_j$ satisfies the Schoenberg-Whitney conditions$^a$ [8]. That is, if only this subset of the data would be given, then a unique interpolating spline function would exist.

In certain applications it may be more appropriate to use other kinds of error functions. For instance, for the fitting of uncertain data, the $\ell_1$ norm of the error vector

$$ (FA_1 : ) \quad \sum_{j=0}^{n} |f(t_j) - p_j| = \| (f(t_j) - p_j)_{j=0,\ldots,n} \|_1 \to \text{Min} \tag{4} $$

gives better results, as the approximating spline function simply ignores so-called ‘outliers’ [2]. This approximation will be called the functional $\ell_1$ approximation problem $FA_1$. On the other hand, in order to minimize the global error, one may prefer to minimize the $\ell_\infty$ norm of the error vector,

$$ (FA_\infty : ) \quad \max_{j=0,\ldots,n} |f(t_j) - p_j| = \| (f(t_j) - p_j)_{j=0,\ldots,n} \|_\infty \to \text{Min} \tag{5} $$

These leads to the functional $\ell_\infty$ approximation problem $FA_\infty$.

As observed by Barrodale and Young in 1966, both approximation tasks can easily be formulated as linear programming (LP) problems, see [3] and the recent article [2]. The problem $FA_1$ is equivalent to the LP problem

$$ (FA_1 : ) \quad \left\{ \begin{array}{l}
\sum_{j=0}^{n} \epsilon_j \to \text{Min} \\
\text{subject to} \quad -\epsilon_j \leq \sum_{i=0}^{m} N_i^d(t_j) c_i - p_j \leq \epsilon_j \quad (j = 0, \ldots, n) 
\end{array} \right\} \tag{6} $$

with the $m + n + 2$ variables $(\epsilon_j)_{j=0,\ldots,n}$ and $(c_i)_{i=0,\ldots,m}$. Similarly, the problem $FA_\infty$ is equivalent to the LP problem

$$ (FA_\infty : ) \quad \left\{ \begin{array}{l}
\epsilon \to \text{Min} \\
\text{subject to} \quad -\epsilon \leq \sum_{i=0}^{m} N_i^d(t_j) c_i - p_j \leq \epsilon \quad (j = 0, \ldots, n) 
\end{array} \right\} \tag{7} $$

$^a$The $m+1$ abscissas $t_{\{(0)\}}, t_{\{(1)\}}, \ldots, t_{\{(m)\}}$ are said to fulfill the Schoenberg-Whitney conditions, if $N_i^d(t_{\{(i)\}}) > 0$ holds for the B-spline functions, $i = 0, \ldots, m$. See [6, §4.8] for more information.
with the \( m + 2 \) variables \( \epsilon \) and \((c_i)_{i=0,...,m}\). A number of algorithms for linear
programming are available, see e.g. [9]. A computationally efficient algorithm for
solving the \( \ell_1 \) approximation problem is described in [4]. The LP formulation of
the \( \ell_\infty \) problem has been used by Esch and Eastman [10] in order to derive an
algorithm for continuous best approximation. For more information on functional
approximation schemes we refer to the survey by Nürnberger [11].

3. Fitting Discrete Data with Parametric Spline Curves

This article is devoted to curve fitting with planar parametric spline curves. A
sequence of data \((p_j)_{j=0,...,n}\) in \( \mathbb{R}^2 \) with an associated sequence of monotonically
increasing parameter values \((t_j)_{j=0,...,n}\) is assumed to be given. There exist several
algorithms for estimating the parameters from the data, see [1, Section 4.4.1]. We
consider a polynomial parametric spline curve of degree \( d \),

\[
x(t) = \sum_{i=0}^{m} N_i^d(t) \mathbf{d}_i, \quad t \in [t_0, t_n],
\]

with the B-spline control points \( \mathbf{d}_i \in \mathbb{R}^2 \), see [1]. The control points \( \mathbf{d}_i \in \mathbb{R}^2 \) are to
be chosen so that the spline curve approximates the given data. As in the previous
section we assume that the knots of the approximating spline curve are already
known.

The B–spline control points \( \mathbf{d}_i \) are to be chosen so that the spline curve (8)
approximates the given data. Similar to the functional case, it is a very popular
approach [1] to compute the unknown spline coefficients (control points) by mini-
mizing the \( \ell_2 \) norm of the error vector,

\[
(PA_2) \quad \sqrt{\sum_{j=0}^{n} |x(t_j) - p_j|^2} = \|x(t_j) - p_j\|_{\ell_2} \to \text{Min},
\]

with the Euclidean norm \( |\mathbf{y}| = \sqrt{y_1^2 + y_2^2} \). This will be called the parametric \( \ell_2 \)
approximation problem \( PA_2 \). It splits into two separate \( \ell_2 \) problems of the type \( FA_2 \)
for the coordinates of the curve. As a consequence, the control points \( (\mathbf{d}_i)_{i=0,...,m}\)
can easily be computed by solving a banded \((m + 1) \times (m + 1)\) system of linear
equations with two right–hand sides. A unique solution of this system can be shown
to exist if and only if a subset of the abscissas \((t_j)_{j=0,...,n}\) satisfies the Schoen–
Whitney conditions, see [8].

Throughout this paper we will assume that the parameter values \((t_j)_{j=0,...,n}\) are
kept constant, they are not subject to optimization. Consequently, the curve (8)
approximates both the given data and their parameterization. Hence the error
vectors \( x(t_j) - p_j \) are generally not perpendicular to the curve tangent at
\( t = t_j \), cf. Figure 3 (b). In order to improve the result of the approximation, by making the
error vectors ‘more orthogonal’ to the curve, one may modify the parameter values
with the help of the method of parameter correction. For any details the reader is
referred to [1, Section 4.4.3].
In the sequel we need to distinguish between various norms. We denote with
\[
\| (y_j)_{j=0, \ldots, n} \|_p \quad \text{for} \quad p \in \{1, 2, \infty\}
\]
the $\ell_p$ norm of a vector from $\mathbb{R}^{n+1}$. On the other hand, we will use the abbreviation $\| y \|$ for the Euclidean norm of a vector $y = (y_1, y_2) \in \mathbb{R}^2$.

The remainder of this section is organized as follows. Firstly we introduce the parametric analogues of the $\ell_1$ and $\ell_\infty$ approximation problems $PA_1, PA_\infty$. Their solutions are shown to form a convex compact set. However, generally no uniqueness of the solution can be expected. Therefore, we propose to modify the original problems in order to get a unique solution.

3.1. **Parametric $\ell_1$ and $\ell_\infty$ Curve Fitting**

Whereas the $\ell_2$ approximation of scattered data is particularly easy to compute, in certain applications it may be advantageous to use other kinds of error functions. For example, if one wants to fit a curve to uncertain data it is more appropriate to compute the control points by minimizing the $\ell_1$ norm of the error vector,

\[
(\text{PA}_1 : ) \quad \sum_{j=0}^{n} \| x(t_j) - p_j \| = \| x(t_j) - p_j \|_{j=0, \ldots, n} \|_1 \to \text{Min},
\]

This task will be called the parametric $\ell_1$ approximation problem $PA_1$. On the other hand, if the global error is to be minimized, one will prefer for compute the control points by minimizing the $\ell_\infty$ norm of the error vector,

\[
(\text{PA}_\infty : ) \quad \max_{j=0, \ldots, n} \| x(t_j) - p_j \| = \| x(t_j) - p_j \|_{j=0, \ldots, n} \|_\infty \to \text{Min},
\]

which leads to the parametric $\ell_\infty$ approximation problem $PA_\infty$. Unlike the $\ell_2$ problem $PA_2$, however, the parametric $\ell_1$ and $\ell_\infty$ approximation problems do not split into separate problems for the coordinates. Thus, in the parametric case, the exact solution of $PA_1$ and $PA_\infty$ cannot be found by linear programming. In Section 4 we describe a linearization technique which leads to an approximate solution.

With each spline curve (8) one may associate the point $D = (d_0, d_1, \ldots, d_m) \in \mathbb{R}^{2m+2}$, simply by collecting the coordinates of the control points.

**Proposition 1.** If a subset of the parameters $(t_j)_{j=0, \ldots, n}$ satisfies the Schoenberg-Whitney conditions, then the solutions $D$ to each of the approximation problems $PA_1$ and $PA_\infty$ form a compact convex subset of $\mathbb{R}^{2m+2}$.

**Proof. 1.) Convexity of the error functions.** Consider two B-spline curves $x(t)$ and $x^*(t)$ (see (8)) with control points $D$ and $D^*$. Then we have for $0 \leq s \leq 1$

\[
\| (1-s) x(t_j) + s x^*(t_j) - p_j |_{j=0, \ldots, n} \|_p \\
\leq (1-s) \| x(t_j) - p_j |_{j=0, \ldots, n} \| + s \| x^*(t_j) - p_j |_{j=0, \ldots, n} \|_p \\
\leq (1-s) \| x(t_j) - p_j |_{j=0, \ldots, n} \| + s \| x^*(t_j) - p_j |_{j=0, \ldots, n} \|_p.
\]

\[
(13)
\]
Thus, both error functions \( ||\cdots||_p, p \in \{1, \infty\} \), are convex continuous functions of the control points.

2.) **Compactness.** Let \((t_j^{(i)})_{i=0, \ldots, m}\) be a sub-sequence of the associated parameters \((t_j)_{j=0, \ldots, n}\) which satisfies the Schoenberg-Whitney conditions. Consequently, the matrix \( N = (N^{(i)}(t_j^{(k)}))_{i, k=0, \ldots, m} \) is invertible. Hence one may uniquely describe the spline curve (8) by its points \( x_i = x(t_i^{(i)}), i = 0, \ldots, m, \) as

\[
D = N^{-1} \cdot X \quad \text{with} \quad X = (x_i)_{i=0, \ldots, m}
\]

(14)

holds. Let \( B \) be the value of the error function \( ||\cdots||_1 \) or \( ||\cdots||_\infty \) for an arbitrary but fixed spline curve \( x^* \). Hence, the solutions to the problems \( PA_1, PA_\infty \), satisfy

\[
B \geq \left\| \begin{bmatrix} x(t_j) - p_j \\ j=0, \ldots, n \end{bmatrix} \right\|_p \geq \left\| \begin{bmatrix} x_i - p_j^{(i)} \\ i=0, \ldots, m \end{bmatrix} \right\|_p.
\]

(15)

That is, the points \( x_i \) of the solutions to \( PA_1, PA_\infty \) are contained within the bounded set

\[
\Omega = \{ X \mid \left\| \begin{bmatrix} x_i - p_j^{(i)} \\ i=0, \ldots, m \end{bmatrix} \right\|_p \leq B \}.
\]

(16)

Owing to \( D = N^{-1} \cdot X \), the control points of the solutions to \( PA_1, PA_\infty \) are contained within the bounded set \( N^{-1} \cdot \Omega \). Thus, both error functions have their global minimum within the compact set \( N^{-1} \cdot \Omega \), as they are convex and continuous. Therefore, the set of optimal points \( D \) is convex and compact. \( \square \)

**Remarks.** 1.) Generally, the approximation problems \( PA_1 \) and \( PA_\infty \) do not have a unique solution. Two examples for this fact are shown in Figure 1. Both examples are B-spline curves of degree \( d = 1 \), i.e., polygonal lines. Whereas the boxes

represent their control points, the crosses and the circles mark the data \( p_j \) and the corresponding points \( x(t_j) \) on the curves, respectively. The Figures 1a and b show two solutions to the approximation problems \( PA_1 \) and \( PA_\infty \) for certain data. The solutions have been drawn as a black and as a grey curve. For many configurations of data \( (p_j)_{j=0, \ldots, n} \) and associated parameters \( (t_j)_{j=0, \ldots, n} \), the problems \( PA_1 \) and \( PA_\infty \) can be expected to have unique solutions. If the data contains ‘outliers’, however, then the value of the objective function of \( PA_\infty \) (maximum distance) is only determined by the spline curve in the region of the ‘outliers’. The remainder of the curve can then vary within a certain region, cf. Figure 1b.

2.) Clearly, computing the inverse \( N^{-1} \) might easily lead to numerical problems.
This inverse, however, is only required theoretically, in order to prove the compactness of the set of solutions. The solutions can be computed via linear and quadratic programming as described in Section 4.

3.2. Modified Parametric $\ell_1$ and $\ell_\infty$ Curve Fitting

In order to guarantee the uniqueness of the solution, we modify the parametric $\ell_1$ and $\ell_\infty$ approximation problems. The control points of the approximating spline curve (8) are found from

\[
(MPA_p) \quad \|\mathbf{x}(t_j) - \mathbf{p}_j\|_{\ell_p} \rightarrow \text{Min}
\]

subject to \( \|\mathbf{x}(t_j) - \mathbf{p}_j\|_2 \leq \epsilon \)

where \( \epsilon = \min \{ \|\mathbf{x}(t_j) - \mathbf{p}_j\|_2 \mid \mathbf{D} \in \mathbb{R}^{2m+2} \} \)

\( (p \in \{1, \infty\}) \). That is, among the solutions of the approximation problems PA$_1$ and PA$_\infty$ we choose the one which has the minimum $\ell_2$ error. Clearly, instead of the $\ell_2$ error function one might also use another quadratic function of the control points to pick the unique solution, e.g., a linearized ‘energy’ functional, see [1, Section 3.6]. According to our numerical experiences, however, the choice of the quadratic function is not so important, as most data will lead to unique solutions to the problems PA$_p$.

**Proposition 2.** If a subset of the parameters \((t_j)_{j=0,...,n}\) satisfies the Schoenberg-Whitney conditions, then the modified approximation problems MPA$_1$ and MPA$_\infty$ have unique solutions.

**Proof.** According to Proposition 1, the feasible regions of MPA$_1$ and MPA$_\infty$ in the control point space \(\{ \mathbf{D} = (\mathbf{d}_i)_{i=0,...,m} \mid \mathbf{D} \in \mathbb{R}^{2m+2} \} \) form a convex compact set. On the other hand, as a subset of the parameters \((t_j)_{j=0,...,n}\) is assumed to satisfy the Schoenberg-Whitney conditions, the least-squares sum

\[
\|\|\mathbf{x}(t_j) - \mathbf{p}_j\|_{j=0,...,n}\|_2
\]

(18) can be shown to be a strongly convex function of the control points, see [8]. Hence, a unique solution to MPA$_1$ and MPA$_\infty$ exists. \( \square \)

Thus, the modified approximation problems MPA$_1$ and MPA$_\infty$ are a well-defined formulation for the task of parametric $\ell_1$ and $\ell_\infty$ curve fitting. In the remainder of this paper we discuss the computational aspects of these problems.

4. Computation of Parametric $\ell_1$ and $\ell_\infty$ Approximants

In the functional case, the $\ell_1$ and $\ell_\infty$ approximation problems could be formulated as linear programming problems. In addition, the analogues of the modified $\ell_1$ and $\ell_\infty$ approximation could simply be formulated as quadratic programming problems (that is, a quadratic objective function is to be minimized subject to linear equality and inequality constraints). For both types of programming problems, a number of
powerful solvers exists [9]. According to theoretical results, the exact solution can be found in finite time.

In the parametric case, however, the situation becomes more difficult. In the sequel we present a linearization technique which can be used in order to compute approximate solutions to both modified parametric approximation problems $\text{MPA}_1$ and $\text{MPA}_\infty$.

4.1. **Linearized Euclidean Norm**

In order to compute an approximate solution to the approximation problems $\text{MPA}_1$ and $\text{MPA}_\infty$ we introduce a linearization of the Euclidean norm. We choose $p+1$ angles $(\phi_k)_{k=0,\ldots,p}$ with $0 \leq \phi_0 < \phi_1 < \ldots < \phi_p < 2\pi$. With these angles we associate the $p+1$ unit vectors

$$\vec{u}_k = \begin{pmatrix} \cos \phi_k \\ \sin \phi_k \end{pmatrix}, \quad k = 0, \ldots, p.$$  \hspace{1cm} (19)

We assume that the oriented angle between neighbouring unit vectors is always smaller than $\pi$, i.e. $\phi_k - \phi_{k-1} < \pi$ ($k = 1, \ldots, p$) and $2\pi + \phi_0 - \phi_p < \pi$. The dot $\cdot$ stands for the inner product of two vectors. Let

$$[y] = \max_{k=0,\ldots,p} \left( \frac{\vec{u}_k + \vec{u}_{(k+1)\mod p} \cdot y}{1 + \vec{u}_k \cdot \vec{u}_{(k+1)\mod p}} \right)$$  \hspace{1cm} (20)

The function $[\cdot] : \mathbb{R}^2 \to \mathbb{R} : y \mapsto [y]$ is a piecewise linear function of the components of $y$. It approximates the Euclidean norm $\|\cdot\|$.

**Lemma 3.** For all vectors $y \in \mathbb{R}^2$, the linearized norm $[\cdot]$ satisfies the inequality

$$C \|y\| \leq [y] \leq \|y\|$$  \hspace{1cm} (21)

with the constant $C = \frac{1}{\pi} \min_{k=0,\ldots,p} \|\vec{u}_k + \vec{u}_{(k+1)\mod p}\|$.

**Proof.** Consider the points $y \in \mathbb{R}^2$ with $[y] = 1$. These points form a polygon with the vertices $(\vec{u}_k)_{k=0,\ldots,p}$, see Figure 2. On the one hand, the vertices are points on the Euclidean unit circle. On the other hand, the maximum radius of the inscribed circles with center $O$ equals $C$. \hfill $\square$

![Fig 2. Linearized Euclidean norm](image)

**Remark.** By using the linearized norm (20), the points satisfying $[y] = 1$ form an inscribed polygon to the unit circle. Alternatively, one might also use the linearized
norm $\|y\|^* = \max_{k=0,\ldots,p}(u_k \cdot y)$ instead. Here, the points satisfying $\|y\|^* = 1$ form a *circumscribed* polygon to the unit circle. Again, one may easily find an inequality similar to (21).

If we increase the number of angles $(\phi_k)_{k=0,\ldots,p}$ so that the angles between neighbouring unit vectors $\hat{u}_k$ and $\hat{u}_{(k+1)\text{mod} \, p}$ tend to zero, then the value of the linearized norm $\|y\|$ converges to the Euclidean norm $\|y\|$. The constant $C$ measures the accuracy of the linearized Euclidean norm, see Lemma 3. We will quantify the accuracy by the value of $C$ in % . The bigger the value of $C$ is, the higher the accuracy gets.

### 4.2. Linearized Parametric $\ell_1$ and $\ell_\infty$ Curve Fitting

With the help of the linearized Euclidean norm we introduce linearized versions of the approximation problems PA$_1$ and PA$_\infty$. We simply replace the Euclidean norm $\|\| \|$ with its linearized version $\|\|_\cdot$. This leads to the linearized parametric $\ell_1$ approximation problem LPA$_1$,

$$\text{(LPA}_1 : \sum_{j=0}^{n} \| \mathbf{x}(t_j) - \mathbf{p}_j \|_1 = \| \mathbf{x}(t_j) - \mathbf{p}_j \|_{j=0,\ldots,n} \rightarrow \text{Min}, \quad (22)$$

and to the linearized parametric $\ell_\infty$ approximation problem LPA$_\infty$,

$$\text{(LPA}_\infty : \max_{j=0,\ldots,n} \| \mathbf{x}(t_j) - \mathbf{p}_j \|_\infty = \| \mathbf{x}(t_j) - \mathbf{p}_j \|_{j=0,\ldots,n} \rightarrow \text{Min}. \quad (23)$$

Note that both tasks are *linear programming problems*, as the linearized norm $\|\|$ is found as the maximum of a number of inner products. For instance, the first linearized problem may equivalently be formulated as

$$\text{(LPA}_\infty : \begin{cases} \sum_{j=0}^{n} \epsilon_j \rightarrow \text{Min} \\
\text{subject to } \sum_{i=0}^{n} N(t_j) d_i - \epsilon_j \leq \frac{\hat{u}_k + \hat{u}_{(k+1)\text{mod} \, p}}{1 + \hat{u}_k \cdot \hat{u}_{(k+1)\text{mod} \, p}} \leq \epsilon_j \\
\text{for } j = 0,\ldots,n; \, k = 0,\ldots,p. \quad (24)$$

The properties of the solutions to the linearized problems are analogous to those of the original problems, cf. Proposition 1.

**Proposition 4.** If a subset of the parameters $(t_j)_{j=0,\ldots,n}$ satisfies the Schoenberg–Whitney conditions, then the solutions $\mathbf{D}$ of the linearized approximation problems LPA$_1$ and LPA$_\infty$ form a compact convex subset of $\mathbb{R}^{2m+2}$.

As the proof is very similar to Proposition 1, it is omitted here.

In addition to the above proposition, the following fact can be shown. If we increase the number of angles $(\phi_k)_{k=0,\ldots,p}$ which are used for defining the linearized norm $\|\|$ (see (20)) so that the angles between neighbouring unit vectors $\hat{u}_k$ and
\( \mathbf{\bar{u}}_{(k+1) \mod p} \) tend to zero, then the solutions of the linearized approximation problems \( \text{LPA}_1 \) and \( \text{LPA}_\infty \) converge to the solutions of the original problems \( \text{PA}_1 \) and \( \text{PA}_\infty \). Once again, the linearized approximation problems \( \text{LPA}_1 \) and \( \text{LPA}_\infty \) do not have a unique solution in the general case. Similar to Figure 1 one may easily construct examples for non-uniqueness.

### 4.3. Modified Linearized Parametric \( \ell_1 \) and \( \ell_\infty \) Curve Fitting

In order to guarantee the uniqueness of the solution, we modify the parametric \( \ell_1 \) and \( \ell_\infty \) approximation problems. The control points of the approximating spline curve (8) are found from

\[
\text{MLPA}_p : \quad \| [ \mathbf{x}(t_j) - \mathbf{p}_j ]_{j=0,\ldots,n} \|_2 \to \text{Min} \\
\text{subject to} \quad \| [ \mathbf{x}(t_j) - \mathbf{p}_j ]_{j=0,\ldots,n} \|_p \leq \epsilon \\
\text{where} \quad \epsilon = \text{min} \{ \| [ \mathbf{z}(t_j) - \mathbf{p}_j ]_{j=0,\ldots,n} \|_p | \mathbf{\bar{D}} \in \mathbb{R}^{2m+2} \}
\]

\((p \in \{1, \infty\})\). That is, among the solutions of the linearized approximation problems \( \text{LPA}_1 \) and \( \text{LPA}_\infty \), we choose the one which has the minimum \( \ell_2 \) error.

**Proposition 5.** If a subset of the parameters \((t_j)_{j=0,\ldots,n}\) satisfies the Schoenberg-Whitney conditions, then the two modified approximation problems \( \text{MLPA}_1 \) and \( \text{MLPA}_\infty \) have unique solutions.

The proof is analogous to that of Proposition 2.

In addition to uniqueness of the solution, the following fact can be observed. If we increase the number of angles \((\phi_k)_{k=0,\ldots,p}\) which are used for defining the linearized norm \([\cdot]\) (see (20)) so that the angles between neighboring unit vectors \( \mathbf{\bar{u}}_k \) and \( \mathbf{\bar{u}}_{(k+1) \mod p} \) tend to zero, then the solutions of the linearized approximation problems \( \text{LPA}_1 \) and \( \text{LPA}_\infty \) converge to the solutions of the original problems \( \text{PA}_1 \) and \( \text{PA}_\infty \). On the one hand, the modified approximation problems \( \text{MLPA}_1 \) and \( \text{MLPA}_\infty \) are well-defined linearized formulation for the task of parametric \( \ell_1 \) and \( \ell_\infty \) curve fitting. On the other hand, the solution is easy to compute. In the first step one has to solve a linear programming problem in order to find the value of the error function. In the second step, the final solution is found by solving a quadratic programming problem. That is, a quadratic objective function is minimized subject to linear inequality constraints. Both optimization problems are standard problems in optimization, see [9].

### 4.4. Implementation

Our implementation is based on the LOQO package [12] which is able to handle both linear and quadratic programming problems. The number of unknowns depends on the type of the problem \( \text{MLPA}_1 \) or \( \text{MLPA}_\infty \). For computing the linearized \( \ell_1 \) approximation we have to use one error bound for each data point, as the objective function is the sum of all these error bounds. That is, we get optimization problems with \( 2(m+1) + (n+1) \) unknowns, where \( m \) and \( n \) are the number of control
points and the number of data, respectively. For computing the linearized $\ell_\infty$ approximation, by contrast, we need to solve optimization problems with only $2(m+1)+1$ unknowns.

The number of linear inequalities depends on the number of the data and on the number $p$ of the angles $(\phi_k)_{k=0, \ldots, p}$ which are used in the linearization step, see (20). We get a system of $(n+1)(p+1)$ linear inequalities. The matrix of this system is sparse, as the B–spline functions are locally supported.

The accuracy of the solution depends on the constant on the left-hand side of the inequality from Lemma 3. In order to generate suitable angles $(\phi_k)_{k=0, \ldots, p}$ for the linearized Euclidean norm we have tested two different strategies:

1.) If one chooses simply a uniform distribution of the angles, $\phi_k = \frac{2k}{p+1} \pi$, Lemma 3 gives the constant $C = \frac{1}{2} \sqrt{2 + 2 \cos \frac{2\pi}{p+1}}$. That is, if we increase the number $p$ of angles $\phi_k$, then we increase the accuracy of the solutions to the parametric modified $\ell_1$ and $\ell_\infty$ approximation problems. This, however, leads to large numbers of inequalities.

2.) In order to overcome this difficulty, one may use an adaptive refinement of the angles $\phi_k$ instead. One starts with a relatively small number of angles and computes an initial solution. With the help of this solution, the angle sequences $(\phi_k)$ are refined by inserting new angles where this is necessary. The refinement is based on the directions of the error vectors $\mathbf{x}(t_j) - \mathbf{p}_j$. Note that this step leads to individual sequences of angles for each one of the given points $\mathbf{p}_j$. Iterating this procedure a few times gives the final solution. As an advantage of this strategy, one may combine it with the method of parameter correction (see [1, Section 4.4.3]). Thus, one could simultaneously optimize both the parameterization of the data and the linearization of the Euclidean norm.

According to our numerical experiments, it is sufficient to use the first strategy for computing solutions with an accuracy (of the approximation to the Euclidean norm) of up to $C = 96\%$, cf. Lemma 3. In order to find highly accurate solutions, however, the second strategy should be preferred, as otherwise the number of inequalities gets too big.

In order to discuss the dependency of computing times and required computer memory from the number of data, we computed a series of approximations to data sets with different data sizes $m$, ranging from 100 to 1200. The number of control points was approximately $n = \frac{1}{2}m$. As indicated by these experiments, the computing time and the required amount of computing time for the optimization grow nearly linearly with the number of data. For instance, in order to fit a parametric spline curve with 200 control points to 600 data, the required optimization time was in the order of 35 seconds. Of course, computing the linearized $\ell_1$ and $\ell_\infty$ approximation is more expensive than the traditional $\ell_2$ fit. The asymptotic behaviour of the computing times, however, seems to be the same.
4.5. Examples

As a first example we approximated 71 data with a centripetal parameterization (see [1]) with a cubic B-spline curve with 26 control points (23 segments). The data contains some 'outliers' and is irregularly distributed. The solutions to the approximation problems MLPA_1 and MLPA_∞ are shown in Figure 3a and c. The solutions have been computed with \( p = 11 \) and by choosing a uniform distribution of the angles \( \phi_k \). This leads to the accuracy \( C = 96\% \) of the linearized norm. In order to compare the result, the \( \ell_2 \) approximation of the data has been computed too, see Figure 3b. The first line of the Figure shows the data (marked by the crosses), the approximating spline curves with their control polygons, and the error vectors (dotted). In the second line, the lengths of the error vectors for the three approximating curves have been plotted. Table 1 shows the numerical values of the three error norms for the three spline curves.

Obviously, the approximating spline curves behave in very different ways. The \( \ell_1 \) approximation \((a,d)\) simply ignores 'outliers', i.e., data with relatively large distance from the remaining points. The overall length of the error vectors is minimized at the cost of the error at the 'outliers'. Consequently, the overall length of the error
vectors is relatively small. The maximum distance error, however, gets quite large. The $\ell_\infty$ approximation (c.f.), by contrast, minimizes the maximum distance error. This is achieved at the cost of the overall error that is visualized by the area of the plot in Figure 3f. As a consequence, the resulting spline curve oscillates a lot. Finally, the $\ell_2$ approximation (b,e) can be seen as a blend of the two extreme cases. It does not ignore the ‘outliers’, but they have less influence to the curve than in the $\ell_\infty$ case.

Another example is presented in Figure 4. The data have been sampled from a ‘zig-zag’ shape. Again, the figure shows only the $\ell_1$, $\ell_2$, and $\ell_\infty$ approximation. One may observe similar effects as in the first example. Note the differences at the corners of the data!

![Fig. 4. Solutions of the approximation problems MLPA_1 (left), MLPA_\infty (right), and the $\ell_2$-approximation (middle) for ‘zig-zag’ shaped data.](image)

### 4.6. Comparison of the error functions

As observed in our experiments, if the data has no ‘outliers’, then in many cases there is only little difference between the $\ell_2$ and the $\ell_\infty$ approximation. The computation of the $\ell_\infty$ approximation, however, is much more expensive. Whereas the solution of MLPA_\infty is found by solving a quadratic programming problem, the $\ell_2$ approximation can simply be computed from a system of linear equations. Thus, we cannot recommend to use $\ell_\infty$ in practice. The $\ell_1$ approximation, by contrast, is very well suited for data which contains ‘outliers’ (‘uncertain data’, see [2]). In many examples, the shape of the $\ell_1$ approximation is much better than the shape of the $\ell_2$ fit. Thus, for uncertain data, using the $\ell_1$ approximation is a valuable alternative approach to data fitting, and the additional efforts may well be justified.

<table>
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<th>$\ell_\infty$ error</th>
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<td>0.053</td>
<td>0.21</td>
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<tr>
<td>$\ell_2$</td>
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<tr>
<td>MLPA_\infty</td>
<td>1.84</td>
<td>0.092</td>
<td>0.073</td>
</tr>
</tbody>
</table>
4.7. Spatial Curves and Surfaces

Finally, we give an outline of the generalization of the previously developed linearization method to the case of spatial curves and surfaces. In the planar case, we replace the unit circle of the exact Euclidean norm with an inscribed polygon. This becomes more difficult in the spatial case, as we have to deal with the unit sphere. Unlike the planar case, there is no canonical distribution of an arbitrary number of points on a sphere, and the construction of optimal distributions is a difficult task. See [13] for coordinates of putatively optimal distributions of \( n \) points on the sphere with \( n \leq 130 \). Summing up, the above approach to parametric \( \ell_1 \) and \( \ell_\infty \) can be generalized to spatial B-spline curves and to tensor-product B-spline surfaces, but its implementation gets more complicated.

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References