# Osculating Paraboloids of Second and Third Order

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Dedicated to Professor Dr. J. Hoschek on the occasion of his 60th birthday

**Abstract.** Osculating paraboloids of second order of a surface have been discussed in classical affine differential geometry. We generalize this concept to cubic osculating paraboloids. This yields a visualization of the local properties of a given surface which depend on the derivatives of maximal order four.

Osculating quadric surfaces (and especially osculating paraboloids of second order) which possess a contact of second order with a given surface have been studied in classical affine differential geometry, see for instance [1, 2, 4, 5, 10, 13, 14, 15, 16]. Using such osculating surfaces it is possible to analyze and to visualize the local properties of a surface which depend on the partial derivatives of maximal order three. In the present paper we will generalize this concept by discussing osculating paraboloids of third order.

In the first section of the paper we derive a very technical lemma which relates the Taylor expansion of order 4 of a surface with respect to different (Cartesian or affine) coordinate systems. Based on this lemma we can easily construct all osculating paraboloids of second and third order of a given surface.

The aim of the second section is to summarize the known results on osculating paraboloids of second order which have been derived in the classical literature. The author believes that such a summary is necessary because the related literature is partially difficult to get and no textbook on affine differential geometry contains a similar survey. Based on the cone of B. Su we present a complete affine classification of non–flat surface points.

Cubic osculating paraboloids of a surface are studied in the third section. Their discussion turns out to be decisively more complicated than that of osculating quadrics, but it yields a number of very promising results.

After discussing flat surface points in the fourth section, we outline in a final remark how the discussion of osculating paraboloids can be used in order to give a geometric characterization of contacts of order $n$ between two surfaces for $n \leq 4$.

## 1. Representing a surface by its Taylor expansion

Consider a regular surface $\Phi$ in the neighbourhood of one of its points $\mathbf{O} \in \Phi$. The point $\mathbf{O}$ is chosen as the origin of a Cartesian $xyz$–coordinate–system, and the $xy$–plane of this system is assumed to be the tangential plane of the given surface $\Phi$ at this point. Then, in a neighbourhood of the point $\mathbf{O}$, the given surface $\Phi$ can be described by a function $z = z(x, y)$ over the $xy$–plane. Let this function $z(x, y)$ be at least four times continuously differentiable at the point $\mathbf{O}$, i.e. at $(x_0, y_0) = (0, 0)$. Then the surface can be approximated in a neighbourhood of the point $\mathbf{O}$ by its Taylor expansion of order four,

$$z = \varphi_2(x, y) + \varphi_3(x, y) + \varphi_4(x, y) + \ldots \tag{1}$$

where

$$\begin{aligned}
\varphi_2(x, y) &= f_{2,0}\, x^2 + f_{1,1}\, x\, y + f_{0,2}\, y^2 \quad , \\
\varphi_3(x, y) &= f_{3,0}\, x^3 + f_{2,1}\, x^2\, y + f_{1,2}\, x\, y^2 + f_{0,3}\, y^3 \quad \text{and} \\
\varphi_4(x, y) &= f_{4,0}\, x^4 + f_{3,1}\, x^3\, y + f_{2,2}\, x^2\, y^2 + f_{1,3}\, x\, y^3 + f_{0,4}\, y^4
\end{aligned}$$

with certain real coefficients $f_{i,j} \in \mathbb{R}$ $(i, j \geq 0, i + j \leq 4)$.
Additionally we will study the Taylor expansion of the surface (1) with respect to another system of coordinates. The three vectors

$$\vec{\mathbf{e}}_x = (\,1 \quad 0 \quad 0\,)\,, \quad \vec{\mathbf{e}}_y = (\,0 \quad 1 \quad 0\,)\,, \quad \text{and} \quad \vec{\mathbf{r}}^{(a,b)} = (\,a \quad b \quad 1\,) \tag{2}$$

(for fixed real numbers $a, b \in \mathbb{R}$) are chosen as the three direction vectors of an *affine* coordinate system with origin $\mathbf{O}$. Let $(\,\xi \;\; \eta \;\; \zeta^{(a,b)}\,)^\top$ denote the coordinates with respect to this system. Analogously to (1), the given surface $\Phi$ can be described by a function $\zeta^{(a,b)} = \zeta^{(a,b)}(\xi, \eta)$ over the $\xi\eta$–plane. From (1) we can compute the Taylor expansion of order four of $\zeta^{(a,b)}(\xi, \eta)$ at the point $(\xi_0, \eta_0) = (0, 0)$:

**Lemma 1.** *The coordinate function $\zeta^{(a,b)} = \zeta^{(a,b)}(\xi, \eta)$ of the given surface (1) with respect to the affine coordinate system (2) possesses at the point $\mathbf{O}$, i.e. at $(\xi_0, \eta_0) = (0, 0)$, the Taylor expansion*

$$\zeta^{(a,b)}(\xi, \eta) = \psi_2^{(a,b)}(\xi, \eta) + \psi_3^{(a,b)}(\xi, \eta) + \psi_4^{(a,b)}(\xi, \eta) + \ldots \tag{3}$$

*of order four, where*

$$\psi_2^{(a,b)}(\xi, \eta) = f_{2,0}\, \xi^2 + f_{1,1}\, \xi\, \eta + f_{0,2}\, \eta^2, \tag{4}$$

$$\psi_3^{(a,b)}(\xi, \eta) = t_1(\xi, \eta) \cdot a + t_2(\xi, \eta) \cdot b + t_3(\xi, \eta) \tag{5}$$

*with*

$$t_1(\xi, \eta) = (2\, f_{2,0}\, \xi + f_{1,1}\, \eta) \left(f_{2,0}\, \xi^2 + f_{1,1}\, \xi\, \eta + f_{0,2}\, \eta^2\right),$$

$$
\begin{aligned}
t_2(\xi,\eta) &= (f_{1,1}\,\xi + 2\,f_{0,2}\,\eta)\left(f_{2,0}\,\xi^2 + f_{1,1}\,\xi\,\eta + f_{0,2}\,\eta^2\right),\\
t_3(\xi,\eta) &= f_{3,0}\,\xi^3 + f_{2,1}\xi^2\eta + f_{1,2}\,\xi\,\eta^2 + f_{0,3}\,\eta^3,
\end{aligned}
$$

*and*

$$
\psi_4^{(a,b)}(\xi,\eta) = \begin{pmatrix} a & b & 1 \end{pmatrix}\cdot \underbrace{\begin{pmatrix} k_{1,1}(\xi,\eta) & k_{1,2}(\xi,\eta) & k_{1,3}(\xi,\eta)\\ k_{2,1}(\xi,\eta) & k_{2,2}(\xi,\eta) & k_{2,3}(\xi,\eta)\\ k_{3,1}(\xi,\eta) & k_{3,2}(\xi,\eta) & k_{3,3}(\xi,\eta)\end{pmatrix}}_{=K(\xi,\eta)} \cdot \begin{pmatrix} a\\ b\\ 1\end{pmatrix} \tag{6}
$$

*with*

$$
\begin{aligned}
k_{1,1}(\xi,\eta) &= 5\,{f_{2,0}}^3\,\xi^4 + 10\,f_{1,1}{f_{2,0}}^2\,\xi^3\eta + 6\,{f_{1,1}}^2 f_{2,0}\,\xi^2\eta^2 + 6\,f_{0,2}{f_{2,0}}^2\,\xi^2\eta^2\\
&\quad +6\,f_{2,0}f_{1,1}f_{0,2}\,\xi\,\eta^3 + {f_{1,1}}^3\,\xi\,\eta^3 + f_{2,0}{f_{0,2}}^2\,\eta^4 + {f_{1,1}}^2 f_{0,2}\,\eta^4,\\
k_{1,2}(\xi,\eta) &= k_{2,1}(\xi,\eta) \;=\; \tfrac{5}{2}\,f_{1,1}{f_{2,0}}^2\,\xi^4 + 4\,f_{0,2}{f_{2,0}}^2\,\xi^3\eta\\
&\quad +4\,{f_{1,1}}^2 f_{2,0}\,\xi^3\eta + 9\,f_{2,0}f_{0,2}f_{1,1}\,\xi^2\eta^2 + \tfrac{3}{2}\,{f_{1,1}}^3\,\xi^2\eta^2\\
&\quad +4\,{f_{1,1}}^2 f_{0,2}\,\xi\,\eta^3 + 4\,f_{2,0}{f_{0,2}}^2\,\xi\,\eta^3 + \tfrac{5}{2}\,f_{1,1}{f_{0,2}}^2\,\eta^4,\\
k_{1,3}(\xi,\eta) &= k_{3,1}(\xi,\eta) \;=\; \tfrac{5}{2}\,f_{3,0}f_{2,0}\,\xi^4 + 2\,f_{2,1}f_{2,0}\,\xi^3\eta + 2\,f_{1,1}f_{3,0}\,\xi^3\eta\\
&\quad +\tfrac{3}{2}\,f_{2,0}f_{1,2}\,\xi^2\eta^2 + \tfrac{3}{2}\,f_{1,1}f_{2,1}\,\xi^2\eta^2 + \tfrac{3}{2}\,f_{0,2}f_{3,0}\,\xi^2\eta^2 + f_{1,1}f_{1,2}\,\xi\,\eta^3\\
&\quad +f_{2,0}f_{0,3}\,\xi\,\eta^3 + f_{2,1}f_{0,2}\,\xi\,\eta^3 + \tfrac{1}{2}f_{1,2}f_{0,2}\,\eta^4 + \tfrac{1}{2}f_{1,1}f_{0,3}\,\eta^4,\\
k_{2,2}(\xi,\eta) &= {f_{1,1}}^2 f_{2,0}\,\xi^4 + f_{0,2}{f_{2,0}}^2\,\xi^4 + 6\,f_{0,2}f_{1,1}f_{2,0}\,\xi^3\eta + {f_{1,1}}^3\,\xi^3\eta\\
&\quad +6\,f_{2,0}{f_{0,2}}^2\,\xi^2\eta^2 + 6\,{f_{1,1}}^2 f_{0,2}\,\xi^2\eta^2 + 10\,f_{1,1}{f_{0,2}}^2\,\xi\,\eta^3 + 5\,{f_{0,2}}^3\,\eta^4,\\
k_{2,3}(\xi,\eta) &= k_{3,2}(\xi,\eta) \;=\; \tfrac{1}{2}f_{2,1}f_{2,0}\,\xi^4 + \tfrac{1}{2}f_{1,1}f_{3,0}\,\xi^4 + f_{0,2}f_{3,0}\,\xi^3\eta\\
&\quad +f_{1,2}f_{2,0}\,\xi^3\eta + f_{1,1}f_{2,1}\,\xi^3\eta + \tfrac{3}{2}\,f_{2,0}f_{0,3}\,\xi^2\eta^2 + \tfrac{3}{2}\,f_{1,1}f_{1,2}\,\xi^2\eta^2\\
&\quad +\tfrac{3}{2}\,f_{0,2}f_{2,1}\,\xi^2\eta^2 + 2\,f_{1,1}f_{0,3}\,\xi\,\eta^3 + 2\,f_{1,2}f_{0,2}\,\xi\,\eta^3 + \tfrac{5}{2}\,f_{0,3}f_{0,2}\,\eta^4,\\
k_{3,3}(\xi,\eta) &= f_{4,0}\,\xi^4 + f_{3,1}\,\xi^3\eta + f_{2,2}\,\xi^2\eta^2 + f_{1,3}\,\xi\,\eta^3 + f_{0,4}\,\eta^4.
\end{aligned}
$$

*Proof.* With respect to the Cartesian $xyz$–coordinate–system, the surface represented by the Taylor expansion (3) is described by the parametric representation

$$
\begin{aligned}
F^{(a,b)}(\xi,\eta) &= \xi\cdot\vec{\mathbf{e}}_x + \eta\cdot\vec{\mathbf{e}}_y + \zeta^{(a,b)}(\xi,\eta)\cdot\vec{\mathbf{r}}^{(a,b)}\\
&= \begin{pmatrix} \xi + a\cdot\left(\psi_2^{(a,b)}(\xi,\eta) + \psi_3^{(a,b)}(\xi,\eta) + \psi_4^{(a,b)}(\xi,\eta) + \dots\right)\\ \eta + b\cdot\left(\psi_2^{(a,b)}(\xi,\eta) + \psi_3^{(a,b)}(\xi,\eta) + \psi_4^{(a,b)}(\xi,\eta) + \dots\right)\\ \psi_2^{(a,b)}(\xi,\eta) + \psi_3^{(a,b)}(\xi,\eta) + \psi_4^{(a,b)}(\xi,\eta) + \dots \end{pmatrix}.
\end{aligned} \tag{7}
$$

Resulting from (1), the given surface $\Phi$ has in a neighbourhood of its point $\mathbf{O}$ the implicit equation

$$
z - \varphi_2(x,y) - \varphi_3(x,y) - \varphi_4(x,y) - \ldots = 0. \tag{8}
$$

If the components of the parametric representation (7) are substituted for $x, y$ and $z$ in the implicit equation (8), all terms whose total degree in $\xi$ and $\eta$ is less or equal than 4 vanish. This proves the assertion. □

*Remark.* The coefficients of the Taylor expansion (3) have been generated using the computer algebra system Maple V. The first appendix contains the Maple code of this computation.

For $a = b = 0$ the affine coordinate system (2) becomes the original Cartesian system. In this case, the Taylor expansions (3) and (1) are identical.
Note that the second order terms in (3) do not depend on the parameters $a$ and $b$, i.e., they are independent of the direction vector $\vec{\mathbf{r}}^{(a,b)}$. This fact results from the affine invariance of the Dupin indicatrix $\varphi_2(x,y) = \pm k$ ($k \in \mathbb{R}$, constant).
The Taylor expansion (3) will be the main tool for the discussion of osculating paraboloids of the given surface $\Phi$. Parametric representations of osculating paraboloids of second or third order result from (7) by cancelling the terms whose total degree in $\xi$ and $\eta$ is higher than 2 or 3, respectively.

## 2. Osculating paraboloids of second order

From now on we always assume $\varphi_2(x,y) \not\equiv 0$, i.e., the point $\mathbf{O}$ should be no flat point of the given surface $\Phi$. By cancelling all terms of higher than second order we obtain from (7) the parametric representation

$$P_2^{(a,b)}(\xi,\eta) = \begin{pmatrix} \xi + a \cdot \psi_2^{(a,b)}(\xi,\eta) \\ \eta + b \cdot \psi_2^{(a,b)}(\xi,\eta) \\ \psi_2^{(a,b)}(\xi,\eta) \end{pmatrix}. \qquad (9)$$

It describes the unique osculating paraboloid of second order with the axis direction $\vec{\mathbf{r}}^{(a,b)}$ which has at $\mathbf{O}$ a contact of second order with the given surface $\Phi$. Resulting from Lemma 1, this paraboloid intersects the given surface in the curve described by the implicit equation

$$\psi_3^{(a,b)}(\xi,\eta) + \psi_4^{(a,b)}(\xi,\eta) + \ldots = 0 \qquad (10)$$

in the parameter domain (i.e., in the tangential plane $z = 0$) of (9). In general, the intersection curve has a triple point at $\mathbf{O}$, where the three tangents (which may be complex or multiple) are the null directions of the cubic form $\psi_3^{(a,b)}(\xi,\eta)$. These three tangents will be called the *osculating tangents* of the paraboloid (9). The relationship between the axis direction $\vec{\mathbf{r}}^{(a,b)}$ and the osculating tangents has been studied in classical affine differential geometry, see [1, 2, 4, 5, 10, 14, 15, 16]. Here is an outline of the main results:

### 2.1 The Transon plane of a tangent

We ask for all paraboloids $P_2^{(a,b)}(\xi,\eta)$ (see (9)) which possess the fixed tangent $T = \{\, (\,\mu\cdot\xi_T \quad \mu\cdot\eta_T \quad 0\,)^\top \mid \mu \in \mathbb{R}\,\}$ as an *osculating* tangent ($(\xi_T,\eta_T) \in \mathbb{R}^2 \setminus \{(0,0)\}$). As a necessary condition, the parameters $a$ and $b$ of the axis direction $\vec{\mathbf{r}}^{(a,b)}$ have to satisfy the linear equation

$$\psi_3^{(a,b)}(\xi_T,\eta_T) = t_1(\xi_T,\eta_T)\cdot a + t_2(\xi_T,\eta_T)\cdot b + t_3(\xi_T,\eta_T) = 0, \tag{11}$$

see (5). Thus, provided that this equation does not degenerate, the axes $\mu\cdot\vec{\mathbf{r}}^{(a,b)}$ ($\mu\in\mathbb{R}$) of the osculating paraboloids $P_2^{(a,b)}(\xi,\eta)$ with osculating tangent $T$ span the plane

$$t_1(\xi_T,\eta_T)\cdot x + t_2(\xi_T,\eta_T)\cdot y + t_3(\xi_T,\eta_T)\cdot z = 0. \tag{12}$$

This plane is called the *Transon plane* of the tangent $T$. It degenerates for 3 tangents $T$ at most because $\varphi_2(x,y)\not\equiv 0$ was assumed.
If the tangent $T$ is an asymptotic tangent ($\varphi_2(\xi_T,\eta_T)=0$) at the point **O**, then the Transon plane becomes the tangential plane $z=0$ of the surface $\Phi$, cf. (5). Otherwise, the direction of the line in which the Transon plane (12) and the tangential plane intersect is *conjugated* to that of the tangent $T$ with respect to the Dupin indicatrix of the surface.
Consider the intersection curve of the surface $\Phi$ with an arbitrary plane through the point **O**. As observed already by Transon, its affine normal at **O** (i.e., the axis of it osculating parabola) is situated in the Transon plane of its tangent, provided that the Transon plane does not degenerate [2, p. 128], [16].

## 2.2 The cone of B. Su

Now we consider the system of all Transon planes to the tangents $T$ at the surface point **O**. This system envelops a rational cone through **O**, the cone of B. Su [10]. If this cone does not degenerate to a line or a plane (see below), than it contains the axes directions $\vec{\mathbf{r}}^{(a,b)}$ of all osculating paraboloids $P_2^{(a,b)}(\xi,\eta)$ which possess a (at least) *double* osculating tangent, where the Transon plane of this double tangent has a first order contact with Su's cone along the generating line $\mu\cdot\vec{\mathbf{r}}^{(a,b)}$ ($\mu\in\mathbb{R}$).
A rational parametric representation of Su's cone can be found from

$$C(\lambda,\mu) = \mu\cdot\begin{pmatrix} t_1(1,\lambda)\\ t_2(1,\lambda)\\ t_3(1,\lambda)\end{pmatrix}\times\begin{pmatrix}\partial_\eta t_1(\xi,\eta)\\ \partial_\eta t_2(\xi,\eta)\\ \partial_\eta t_3(\xi,\eta)\end{pmatrix}\Bigg|_{(\xi,\eta)=(1,\lambda)} \qquad \lambda\in\mathbb{R}\cup\{\infty\},\mu\in\mathbb{R} \tag{13}$$

($\partial_\eta = \frac{\partial}{\partial\eta}$). Resulting from this and from (5), Su's cone has at most the algebraic class 3 and order 4. We have the following possibilities which yield an *affine classification of regular surface points* with respect to the geometric properties of Su's cone:

- *Case 1:* The cone of B. Su is of third class (and fourth order). Then the point **O** is either an elliptic or an hyperbolic point of the surface $\Phi$ and Su's cone has a contact of first order with the tangential plane $z=0$ along the two asymptotic tangents. Figure 1 shows Su's cone and a Transon plane in the case of an elliptic point **O**.

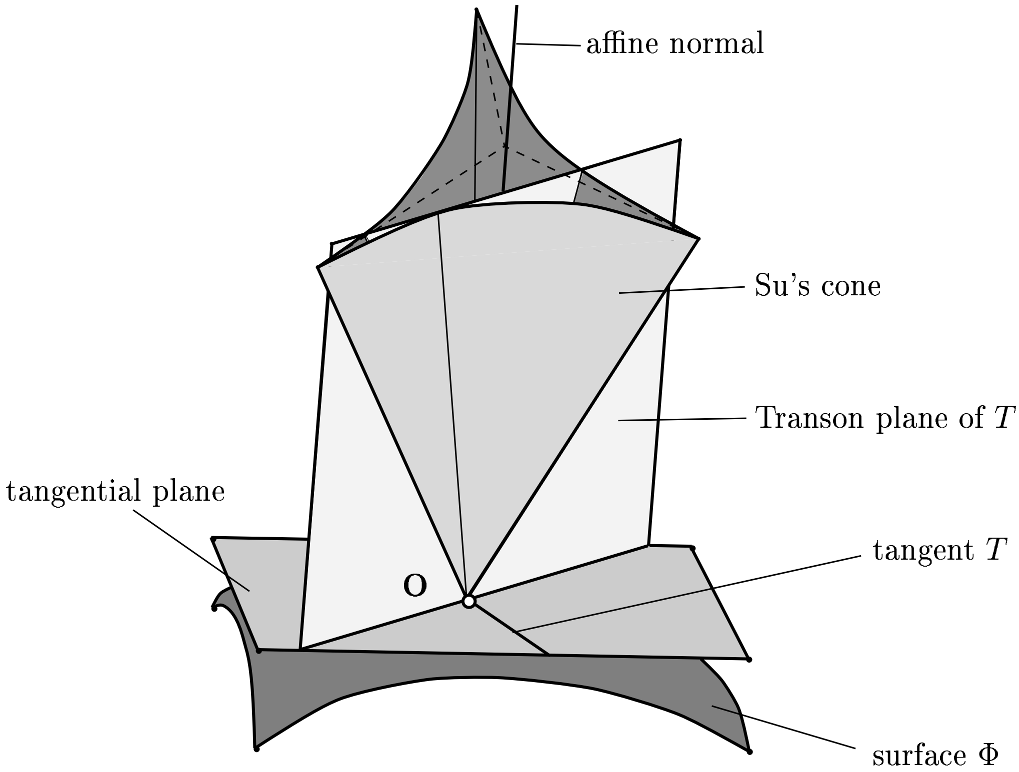


Figure 1: Su's cone and a Transon plane for an elliptic point **O**.

In Case 1, Su's cone possesses three different cuspidal edges. These edges are axes $\mu \cdot \vec{\mathbf{r}}^{(a,b)}$ $(\mu \in \mathbb{R})$ of those osculating paraboloids $P_2^{(a,b)}(\xi,\eta)$ which have a *triple* osculating tangent. The parameters $a, b$ of the axis direction of such an osculating paraboloid fulfill the tree equations

$$\psi_3^{(a,b)}(\xi_D,\eta_D) = \frac{\partial}{\partial\eta}\,\psi_3^{(a,b)}(\xi,\eta)\Big|_{(\xi,\eta)=(\xi_D,\eta_D)} = \frac{\partial^2}{\partial\eta}\,\psi_3^{(a,b)}(\xi,\eta)\Big|_{(\xi,\eta)=(\xi_D,\eta_D)} = 0 \qquad (14)$$

where $(\,\xi_D\ \eta_D\ 0\,)^\top$ is the direction of the triple osculating tangent $((\xi_D,\eta_D) \in \mathbb{R}^2 \setminus (0,0))$. The resulting three linear equations for $a$ and $b$ are solvable if and only if the determinant of the $2\times 3$ coefficient matrix and of the right–hand side vanishes. This yields the cubic equation

$$\begin{aligned} 0 \;=\; & {f_{1,1}}^2 f_{3,0}\,{\xi_D}^3 - f_{0,2}f_{2,0}f_{3,0}\,{\xi_D}^3 - f_{2,0}f_{1,1}f_{2,1}\,{\xi_D}^3 + {f_{2,0}}^2 f_{1,2}\,{\xi_D}^3 \\ & -3\,f_{0,2}f_{2,0}f_{2,1}\,{\xi_D}^2\eta_D + 3\,{f_{2,0}}^2 f_{0,3}\,{\xi_D}^2\eta_D + 3\,f_{0,2}f_{1,1}f_{3,0}\,{\xi_D}^2\eta_D \\ & +3\,f_{2,0}f_{0,3}f_{1,1}\,\xi_D\,{\eta_D}^2 + 3\,{f_{0,2}}^2 f_{3,0}\,\xi_D\,{\eta_D}^2 - 3\,f_{0,2}f_{2,0}f_{1,2}\,\xi_D\,{\eta_D}^2 \\ & -f_{0,2}f_{2,0}f_{0,3}\,{\eta_D}^3 - f_{0,2}f_{1,1}f_{1,2}\,{\eta_D}^3 + f_{0,3}{f_{1,1}}^2\,{\eta_D}^3 + {f_{0,2}}^2 f_{2,1}\,{\eta_D}^3 \end{aligned} \qquad (15)$$

for the three tangents $D_i = \{\,(\,\mu{\cdot}\xi_{D_i}\ \ \mu{\cdot}\eta_{D_i}\ \ 0\,)^\top \mid \mu \in \mathbb{R}\,\}$ $(i = 1,2,3)$ which are triple osculating tangents of osculating paraboloids $P_2^{(a,b)}(\xi,\eta)$. These tangents are pairwise linearly independent. They are called the *Darboux tangents* of the given surface $\Phi$ at $\mathbf{O}$. The three Darboux tangents and the corresponding cuspidal edges of Su's cone are either all real (for elliptic points $\mathbf{O}$) ore one is real and two are conjugated–complex (for hyperbolic points $\mathbf{O}$). The three Transon planes of the Darboux directions intersect in one line, the *affine normal* of the surface. Moreover, the three Darboux tangents are the osculating tangents of the paraboloid $P_2^{(a^*,b^*)}(\xi,\eta)$, whose axis is the affine normal. This normal possesses the direction vector $(\,a^*\;b^*\;1\,)^\top$ with

$$\begin{aligned} a^* &= \frac{-2\,f_{0,2}f_{2,0}f_{1,2}+3\,f_{1,1}f_{2,0}f_{0,3}-6\,f_{0,2}^2f_{3,0}+3\,f_{2,1}f_{1,1}f_{0,2}-f_{1,1}^2f_{1,2}}{\left(4\,f_{0,2}f_{2,0}-f_{1,1}^2\right)^2} \quad \text{and} \\ b^* &= -\frac{6\,f_{2,0}^2f_{0,3}-3\,f_{1,1}f_{2,0}f_{1,2}+2\,f_{2,0}f_{0,2}f_{2,1}+f_{1,1}^2f_{2,1}-3\,f_{3,0}f_{1,1}f_{0,2}}{\left(4\,f_{0,2}f_{2,0}-f_{1,1}^2\right)^2}\,. \end{aligned} \tag{16}$$

- *Case 2:* Su's cone degenerates to a quadratic cone. Then the three cubic polynomials $t_1(\xi,\eta), t_2(\xi,\eta)$ and $t_3(\xi,\eta)$ (see (5)) must have exactly one common null direction $(\,\xi_n\;\eta_n\;0\,)^\top$, because Su's cone is only of class 2 ($(\xi_n,\eta_n)\in\mathbb{R}^2\setminus\{(0,0)\}$). Thus, they possess the common linear factor $(\eta_n\cdot\xi-\xi_n\cdot\eta)$. Resulting from the factorization of $t_1$ and $t_2$, the tangent $N=\{\,(\,\mu{\cdot}\xi_n\;\;\mu{\cdot}\eta_n\;\;0\,)^\top\mid\mu\in\mathbb{R}\,\}$ is one of the asymptotic tangents of the surface $\Phi$, where $\mathbf{O}$ is a hyperbolic point. In the other asymptotic tangent, the quadratic cone has a first order contact with the tangential plane $z=0$. In this case, all osculating paraboloids $P_2^{(a,b)}(\xi,\eta)$ possess the osculating tangent $N$, because $\psi_3^{(a,b)}(\xi_n,\eta_n)=0$ holds for all $a,b\in\mathbb{R}$. Exactly one osculating paraboloid has $N$ as triple osculating tangent, its axis is the affine normal of the surface. The tangent $N$ is the unique Darboux direction.

- *Case 3:* The Transon planes form a pencil of planes through a line $P=\{\,(\,\mu{\cdot}p\;\;\mu{\cdot}q\;\;\mu{\cdot}r\,)^\top\mid\mu\in\mathbb{R}\,\}$ where this line is no tangent of the surface $\Phi$ ($p,q,r\in\mathbb{R}, \mathrm{r}\neq 0$). Then, the Transon planes have to satisfy the equation

$$t_1(\xi,\eta)\cdot p+t_2(\xi,\eta)\cdot q+t_3(\xi,\eta)\cdot r=0 \tag{17}$$

for *all* $\xi,\eta\in\mathbb{R}$. Thus, the polynomial $t_3(\xi,\eta)$ can be represented as

$$t_3(\xi,\eta)=-\frac{1}{r}\left(f_{0,2}\eta^2+f_{1,1}\xi\,\eta+f_{2,0}\xi^2\right)\left(2\,q\,\eta\,f_{0,2}+p\,f_{1,1}\eta+2\,p\,\xi\,f_{2,0}+q\,\xi\,f_{1,1}\right). \tag{18}$$

In this case, the osculating paraboloid with axis direction $\vec{\mathbf{r}}^{(a,b)}=(\,\frac{p}{r}\;\frac{q}{r}\;1)$ has a contact of higher than second order with $\Phi$ as $\psi_3^{(a,b)}(\xi,\eta)=0$ holds for all $\xi,\eta\in\mathbb{R}$. All other osculating paraboloids possess the asymptotic tangents as two of their osculating tangents. The given surface has either an elliptic or a hyperbolic point at $\mathbf{O}$.

- *Case 4:* The Transon planes form a pencil of planes through the tangent $P = \{ (\mu{\cdot}p \quad \mu{\cdot}q \quad 0)^\top \mid \mu \in \mathbb{R} \}$ of the given surface $\Phi$ ($(p,q) \in \mathbb{R}^2 \setminus \{(0,0)\}$). Then the three Transon planes have to satisfy the equation

$$
\begin{aligned}
0 &= t_1(\xi,\eta)\cdot p + t_2(\xi,\eta)\cdot q && (19)\\
&= \left(f_{2,0}\xi^2 + f_{1,1}\xi\,\eta + f_{0,2}\eta^2\right)(2\,f_{2,0}\,p\,\xi + f_{1,1}\,p\,\eta + f_{1,1}\,q\,\xi + 2\,f_{0,2}\,q\,\eta)
\end{aligned}
$$

for all $\xi, \eta \in \mathbb{R}$. So we have $2\,f_{2,0}\,p + f_{1,1}\,q = f_{1,1}\,p + 2\,f_{0,2}\,q = 0$ which yields after some calculations $f_{2,0} = q^2\,k$, $f_{1,1} = -2\,p\,q\,k$, and $f_{0,2} = p^2\,k$ with some constant $k \in \mathbb{R}$. Hence, **O** is a parabolic point with the asymptotic tangent $P$. All osculating paraboloids $P_2^{(a,b)}(\xi,\eta)$, whose axes are contained in one Transon plane, are identical, only the parametric representations are different.
An arbitrary tangent $Q = \{ (\mu{\cdot}\xi_q \quad \mu{\cdot}\eta_q \quad 0)^\top \mid \mu \in \mathbb{R} \}$ is a *double* osculating tangent of the osculating paraboloid $P_2^{(a,b)}(\xi,\eta)$ if and only if $\psi_3^{(a,b)}(\xi_q,\eta_q) = \partial_\xi\,\psi_3^{(a,b)}(\xi_q,\eta_q) = \partial_\eta\,\psi_3^{(a,b)}(\xi_q,\eta_q) = 0$ holds ($(\xi_q,\eta_q) \in \mathbb{R}^2 \setminus \{(0,0)\}$). From these equations we obtain the condition

$$
3\,f_{3,0}\,p\,{\xi_q}^2 + f_{2,1}\,q\,{\xi_q}^2 + 2\,f_{2,1}\,p\,\xi_q\,\eta_q + 2\,f_{1,2}\,q\,\xi_q\,\eta_q + f_{1,2}\,p\,{\eta_q}^2 + 3\,f_{0,3}\,q\,{\eta_q}^2 = 0 \quad (20)
$$

for double osculating tangents in Case 4. We have the following four possible sub–cases:

  - *Case 4.1:* The equation (20) has two different solutions which are not the asymptotic direction of the surface. The two solutions can be either both real or conjugate complex. To each solution we get an osculating paraboloid (corresponding to its Transon plane) which possess the line spanned by the null direction of (20) as a double osculating tangent.
  - *Case 4.2:* The equation (20) has two different solutions, the first one is the asymptotic direction of the surface. We obtain one osculating paraboloid (corresponding to the Transon plane of the second solution of (20)) which possesses the line spanned by the second null direction of (20) as double osculating tangent. Moreover, all osculating paraboloids have the asymptotic tangent of the surface as a single osculating tangent.
  - *Case 4.3:* The equation (20) has a double solution, and this solution is not the asymptotic direction. Then we obtain one osculating paraboloid $P_2^{(a,b)}(\xi,\eta)$ which possesses this solution even as a *triple* osculating tangent.
  - *Case 4.4:* The equation (20) has the asymptotic direction as a double solution. In this case, all osculating paraboloids $P_2^{(a,b)}(\xi,\eta)$ possess the asymptotic tangent as a double osculating tangent.

If the condition (20) is fulfilled for all $\xi, \eta \in \mathbb{R}$, then we have

- *Case 5:* All non–degenerated Transon planes are identical. Similar to the previous case, the point **O** is a parabolic point, and the Transon plane intersects the tangential plane in the asymptotic tangent. Moreover we can show that all paraboloids $P_2^{(a,b)}(\xi,\eta)$ with a axis direction $\vec{\mathbf{r}}^{(a,b)}$ contained in the Transon plane are identical, and they have a contact of higher than second order with the given surface $\Phi$. The other osculating paraboloids possess the asymptotic tangent as a *triple* osculating tangent.

The above list gives a complete overview over the geometric properties of Su's cone and over the existence of osculating paraboloids with double or triple osculating tangents for non–flat surface points. Elliptic (hyperbolic ) surface points belong to the cases 1 or 3 (1, 2, or 3), whereas parabolic points always yield the cases 4 or 5. In the cases 3 and 5 hyperosculating paraboloids (i.e., with a contact of higher than second order) exist. The formula for the affine normal of a surface (16) is valid for non–parabolic surface points, but the equation for the Darboux directions (15) is only true for the first two cases.

# 3. Osculating paraboloids of third order

Similar to the previous section, the parametric representation

$$P_3^{(a,b)}(\xi,\eta) = \begin{pmatrix} \xi + a\cdot(\,\psi_2^{(a,b)}(\xi,\eta) + \psi_3^{(a,b)}(\xi,\eta)\,) \\ \eta + b\cdot(\,\psi_2^{(a,b)}(\xi,\eta) + \psi_3^{(a,b)}(\xi,\eta)\,) \\ \psi_2^{(a,b)}(\xi,\eta) + \psi_3^{(a,b)}(\xi,\eta) \end{pmatrix} \tag{21}$$

of the unique osculating paraboloid of third order with the axis direction $\vec{\mathbf{r}}^{(a,b)}$ which has at **O** a contact of third order with the given surface $\Phi$ is obtained from (3) by cancelling all terms of higher than third order. Resulting from Lemma 1, the cubic paraboloid $P_3^{(a,b)}(\xi,\eta)$ intersects the given surface in the curve which is described by the implicit equation

$$\psi_4^{(a,b)}(\xi,\eta) + \ldots = 0 \tag{22}$$

in the parameter domain (i.e., in the tangential plane $z=0$) of $P_3^{(a,b)}(\xi,\eta)$. In general, the intersection curve has a 4–fold point at **O**, where the four tangents (which may be complex or multiple) are the null directions of the quartic form $\psi_4^{(a,b)}(\xi,\eta)$. These four tangents are called the *osculating tangents* of the cubic paraboloid $P_3^{(a,b)}(\xi,\eta)$.

## 3.1 The axes' cone

We will give a similar discussion of the relationship between the axis direction $\vec{\mathbf{r}}^{(a,b)}$ and the osculating tangents as for the quadratic paraboloids $P_2^{(a,b)}(\xi,\eta)$. As first we again ask for all paraboloids $P_3^{(a,b)}(\xi,\eta)$ which possess the fixed tangent $T = \{\,(\,\mu\cdot\xi_T \quad \mu\cdot\eta_T \quad 0\,)^\top \,|$

$\mu \in \mathbb{R}\}$ as an osculating tangent $((\xi_T, \eta_T) \in \mathbb{R}^2 \setminus \{(0,0)\})$. As a necessary condition (which is sufficient iff $\psi_4^{(a,b)}(\xi,\eta) \not\equiv 0$), the parameters $a$ and $b$ of the axis direction $\vec{\mathbf{r}}^{(a,b)}$ have to satisfy the quadratic equation

$$\psi_4^{(a,b)}(\xi_T,\eta_T) = (\,a \;\; b \;\; 1\,) \cdot K(\xi_T,\eta_T) \cdot \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = 0, \tag{23}$$

cf. (6). Thus, provided that this equation does not degenerate, the axes $\mu \cdot \vec{\mathbf{r}}^{(a,b)}$ $(\mu \in \mathbb{R})$ of the cubic osculating paraboloids with the osculating tangent $T$ form the quadratic cone

$$(\,x \;\; y \;\; z\,) \cdot K(\xi_T,\eta_T) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0. \tag{24}$$

This cone will be called the *axes' cone* of the tangent $T$. Of course, it can be imaginary (with the real point $\mathbf{O}$) or even degenerated:

**Proposition 2.** *The axes' cone either degenerates for the asymptotic tangents and for 6 additional tangents $T$ which may be complex, multiple, or equal to the asymptotic tangents, or it degenerates for all tangents of the given surface.*

*Proof.* The determinant of the $3 \times 3$–matrix $K(\xi,\eta)$ factorizes into the product

$$\frac{5}{4}\left(f_{2,0}\xi^2 + f_{1,1}\xi\,\eta + f_{0,2}\eta^2\right)^3 \left(-5\,f_{0,2}f_{2,0}{f_{3,0}}^2\xi^6 + \;[76 \text{ terms}]\; - 5\,f_{0,2}f_{2,0}{f_{0,3}}^2\eta^6\right) \tag{25}$$

where the third factor is a polynomial of total degree 6 in $\xi$ and $\eta$. If this polynomial vanishes identically, then the axes' cone degenerates for all tangents. □

In the case of a parabolic surface point, at least 2 of the 6 additional tangents are the asymptotic tangent, and the remaining 4 additional tangents degenerate to 2 pairs of double tangents.

For any tangent $T$ of the given surface we have its Transon plane (12) and its axes' cone (24). Additionally we can consider the *Moutard quadric* (see [3, §108], [11], or [14]) of the tangent $T$. It is formed by the osculating conics of all planar sections through the given surface with the tangent $T$.† For instance, the Moutard quadric of the tangent

†The Moutard quadric can be said to be the projective analogue of the Meusnier sphere which is well known from Euclidean differential geometry. Note that also an affine analogue exists: the osculating parabolae of all planar sections through the given surface with tangent $T$ form a parabolic cylinder as observed by T. Kubota in 1930 at first [10].

$\{\,(\mu\;\;0\;\;0)^\top \mid \mu \in \mathbb{R}\}$ has the implicit equation

$$
\begin{aligned}
0 &= f_{2,0}^3\, z \\
&- (x\;\;y\;\;z)\underbrace{\begin{pmatrix} f_{2,0}^4 & \frac{1}{2} f_{2,0}^3\, f_{1,1} & \frac{1}{2} f_{2,0}^2\, f_{3,0} \\ \frac{1}{2} f_{2,0}^3\, f_{1,1} & f_{2,0}^3\, f_{0,2} & \frac{1}{2} f_{2,0}\,(f_{2,0}\, f_{2,1} - f_{1,1}\, f_{3,0}) \\ \frac{1}{2} f_{2,0}^2\, f_{3,0} & \frac{1}{2} f_{2,0}\,(f_{2,0}\, f_{2,1} - f_{1,1}\, f_{3,0}) & f_{4,0}\, f_{2,0} - f_{3,0}^2 \end{pmatrix}}_{= M(1,0)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\end{aligned}
\tag{26}
$$

The Transon plane, the axes' cone, and the Moutard quadric of a tangent are related as follows:

**Proposition 3.** *If the tangent $T$ is no asymptotic tangent of the surface $\Phi$ at $\mathbf{O}$, then the infinite conics (i.e., their intersections with the plane at infinity) of the Moutard quadric, of the axes' cone, and of the Transon plane (which is considered as a double line) belong to a pencil of conics. If the axes' cone is non–degenerated, then this pencil contains exactly one other pair of lines. The cross ratio of this line pair with the above three conics is always equal to $\frac{4}{5}$.*

*Proof.* We consider the tangent $T = \{\,(\mu\;0\;0)^\top \mid \mu \in \mathbb{R}\,\}$. The double line at infinity of its Transon plane and the axes' cone generate the pencil of conics

$$
0 = \Big[ t_1(1,0)\, x + t_2(1,0)\, y + t_3(1,0)\, z \Big]^2 - \tau \cdot f_{2,0} \cdot (x \quad y \quad z) \cdot K(1,0) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{27}
$$

with parameter $\tau \in \mathbb{R} \cup \{\infty\}$ in the infinite plane. If it collapses to a system of line pairs (or double lines), then the axes' cone of the tangent $T$ is degenerated. For $\tau = 0$ we obtain the Transon plane, whereas $\tau = \infty$ yields the axes' cone. Moreover, from $\tau = 1$ we get the infinite conic of the Moutard quadric, and $\tau = \frac{4}{5}$ yields a pair of lines. □

The situation in the plane at infinity has been drawn in Figure 2: The four conics from Proposition 3 are shown, where the infinite line of the tangential plane is chosen as the infinite line of the Figure.

With help of the Moutard quadric we can formulate a geometric characterization of the tangents with a degenerated axes' cone:

**Proposition 4.** *The axes' cone (24) of a tangent degenerates into a pair of planes (which may be complex or identical) if and only if the corresponding Moutard quadric is an elliptic or hyperbolic paraboloid, a cylinder, or (in the case of the asymptotic tangents) a pair of planes.*

*Proof.* The Moutard quadric (26) of the tangent $T = \{\,(\mu\;0\;0)^\top \mid \mu \in \mathbb{R}\,\}$ is an elliptic or hyperbolic paraboloid, a cylinder, or a pair of planes, if its intersection with the plane at infinity degenerates to a pair of lines, i.e., if $\det(M(1,0)) = 0$ holds (cf. (26)). On the

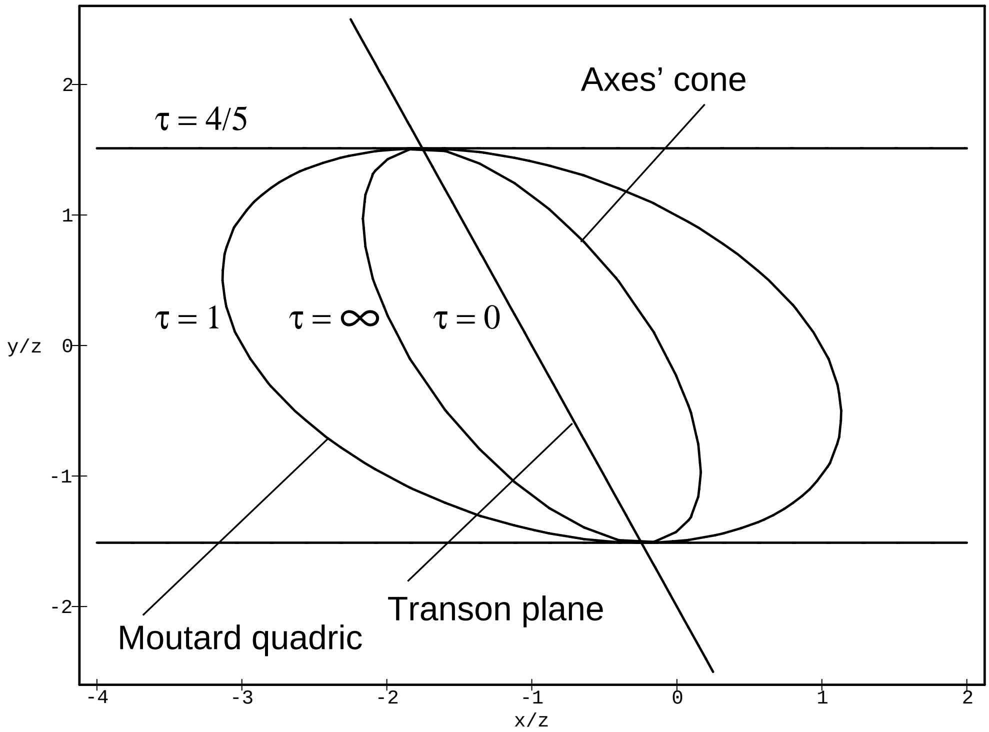


Figure 2: The pencil of conics in the plane at infinity

other hand, the axes' cone degenerates if the equation $\det(K(1,0)) = 0$ is fulfilled. Both conditions turn out to be equivalent. □

The Transon plane of a tangent $T$ intersects the tangential plane of the given surface in another tangent, whose direction is conjugated to that of $T$ with respect to the Dupin indicatrix. Similarly, the intersection of the axes' cone of a tangent $T$ with the tangential plane is already determined by the Dupin indicatrix, i.e., by the second order properties of the given surface:

**Proposition 5.** *If the axes' cone (24) of a tangent $T$ is non–degenerated, then it intersects the tangential plane in two lines $L^{(1)}$ and $L^{(2)}$ (which can be conjugated–complex or identical). For elliptic or hyperbolic surface points, the cross ratio of the line $L^{(1)}$ or $L^{(2)}$ with the tangent $T$ and with the two asymptotic tangents is either equal to $\frac{1}{2} + \frac{1}{10}\sqrt{5}$ or to $\frac{1}{2} - \frac{1}{10}\sqrt{5}$. For parabolic surface points, both lines $L^{(1)}$ and $L^{(2)}$ are identical with the asymptotic tangent.*

*Proof.* Consider the axes' cone (24) of the tangent $T = \{\ (\mu\ 0\ 0)^\top \mid \mu \in \mathbb{R}\ \}$. If this cone is non–degenerated, then it intersects the tangential plane in the pair of lines

$$f_{2,0}{}^2 x^2 + f_{1,1} f_{2,0}\, x\, y + \frac{1}{5} f_{0,2} f_{2,0}\, y^2 + \frac{1}{5} f_{1,1}{}^2 y^2 = 0. \tag{28}$$

If the lines $\{\ (\mu\alpha_1\ \mu\beta_1\ 0)^\top \mid \mu \in \mathbb{R}\ \}$ and $\{\ (\mu\alpha_2\ \mu\beta_2\ 0)^\top \mid \mu \in \mathbb{R}\ \}$ are the asymptotic tangents of the surface $\Phi$ $((\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{C}^2 \setminus \{(0,0)\})$, then we have

$$f_{2,0} := K \cdot \beta_1\beta_2, \quad f_{1,1} := -K \cdot (\alpha_2\,\beta_1 + \alpha_1\,\beta_2) \quad \text{and} \quad f_{0,2} := K \cdot \alpha_1\alpha_2$$

with some constant $K \in \mathbb{R}$. Then, the two lines $L^{(1)}, L^{(2)}$ obtained from (28) possess the direction vectors

$$\vec{\mathbf{l}}^{(1/2)} = \begin{pmatrix} 6\,\alpha_1\,\alpha_2\,\beta_1\,\beta_2 + 2\,{\beta_1}^2{\alpha_2}^2 + 2\,{\alpha_1}^2{\beta_2}^2 \\ \mp\frac{1}{2}\,(\mp 5 + \sqrt{5}) \cdot \left(2\,\alpha_1\,\beta_2 + 3\,\alpha_2\,\beta_1 \pm \sqrt{5}\beta_1\,\alpha_2\right)\beta_1\,\beta_2 \\ 0 \end{pmatrix}.$$

Thus, the cross ratio of the lines $L^{(1)}, L^{(2)}$ with the two asymptotic tangents $A_1, A_2$ and the given tangent $T$ is equal to

$$\mathrm{cr}(A_1, A_2, L^{(1/2)}, T) = \frac{(\alpha_1\,0 - \beta_1\,1) \cdot \left(\vec{\mathbf{l}}_1^{(1/2)}\,\beta_2 - \vec{\mathbf{l}}_2^{(1/2)}\,\alpha_2\right)}{\left(\vec{\mathbf{l}}_1^{(1/2)}\,0 - \vec{\mathbf{l}}_2^{(1/2)}\,1\right) \cdot (\alpha_1\,\beta_2 - \beta_1\,\alpha_2)} = \ldots = \frac{1}{2} \pm \frac{1}{10}\sqrt{5}.$$

This completes the proof. □

Without proof we add still the following two results:

- The infinite point of a tangent $T$ is always polar to the Transon plane (12) of the tangent with respect to the axes' cone (24), provided that neither the axes' cone nor the Transon plane are degenerated.
- Consider the case where Su's cone does not collapse to a line or a plane. If the axes' cone of a tangent $T$ is non–degenerated, then the center point of its intersection with a plane parallel to the tangential plane is situated on the corresponding generating line of Su's cone (i.e., on the line where the Transon plane of $T$ envelops Su's cone), see also Figure 3.

### 3.2 The system of all axes' cones

Now we will discuss the system of all axes' cones to the tangents $T$ of the given surface $\Phi$ at its point $\mathbf{O}$. Similar to the discussion of the system of Transon planes, the axes' cones envelop a cone through the surface point $\mathbf{O}$. We will consider only the most general case, especially we assume that the axes' cone degenerates only for a finite number of tangents:

**Proposition 6.** *For hyperbolic or elliptic surface points, the system of the axes' cones envelops an algebraic cone which has the maximal order 12. This cone intersects the tangential plane $z = 0$ only in the (each 6–fold) asymptotic tangents. For parabolic surface points, the system of the axes' cones envelops an algebraic cone of maximal order 8 and this cone intersects the tangential plane $z = 0$ only in the (8–fold) asymptotic tangent.*

*Proof.* The points on the cone which is enveloped by the system of axes' cones satisfy the two quadratic equations

$$(\,x\ \ y\ \ z\,)\cdot K(\xi,\eta)\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0\quad\text{and}\quad(\,x\ \ y\ \ z\,)\cdot\frac{\partial}{\partial\eta}K(\xi,\eta)\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0. \qquad (29)$$

($\xi,\eta\in\mathbb{R}$). Elimination of the parameters $\xi$ and $\eta$ using the resultant method (see e.g. [6, 17]) yields the equation of an algebraic cone. For elliptic and hyperbolic surface points, the Maple code of the implicitization is presented in the second appendix. As shown using Maple, the intersection of the algebraic cone with the tangential plane $z=0$ factorizes for elliptic and hyperbolic surface points in the product of the equations of the asymptotic tangents to the sixth power. The proof for parabolic points results similarly. $\square$

As a first example, Figure 3 shows the system of the axes' cones, the algebraic cone of order 12 enveloped by this system, and the cone of B. Su for a surface with an elliptic point **O**. In the Figure, the intersections of these cones with the plane $z=1$ have been drawn. Note that some of the axes' cones are imaginary. The centers of the intersections of the axes' cones with the plane $z=1$ are located on the corresponding generator of Su's cone.

The axes' cone (24) possesses also a representation as a cone of algebraic *class* 2. Analogously to the elimination of the parameters $\xi,\eta$ from the equations (29), we can find the algebraic description of the cone of order 12 from the previous Proposition as a *class* cone. It can be shown, that the algebraic class of this cone is not greater than 28 for elliptic or hyperbolic surface points (20 for parabolic points). Unfortunately, Maple V was unable to compute the class equation of this cone in the general case, only in a few examples it was successful. In all these examples, the equation of the cone factorizes and the cone has only the algebraic class 22 for elliptic/hyperbolic surface points and 12 for parabolic points. So the author conjectures that the algebraic class of the cone enveloped by the system of axes' cones is generally equal to 22 or to 12 for elliptic and hyperbolic or for parabolic points, respectively.

In the general case, the cone of algebraic order 12 enveloped by the system of the axes' cones is formed by the axes $\mu\cdot\vec{\mathbf{r}}^{(a,b)}$ ($\mu\in\mathbb{R}$) of the cubic paraboloids $P_3^{(a,b)}(\xi,\eta)$ which have a (at least) double osculating tangent. (To any tangent $T$ generally correspond four cubic paraboloids $P_3^{(a,b)}(\xi,\eta)$ (which may be complex) with $T$ as a double osculating tangent.) The singularities of this cone (e.g., cuspidal edges or double lines) correspond to cubic paraboloids with multiple osculating tangents (e.g., a triple osculating tangent or a pair of double osculating tangents). Of course, the discussion of the geometric properties of the cone and of its singularities is very difficult. At this point we cannot present any interesting further results.

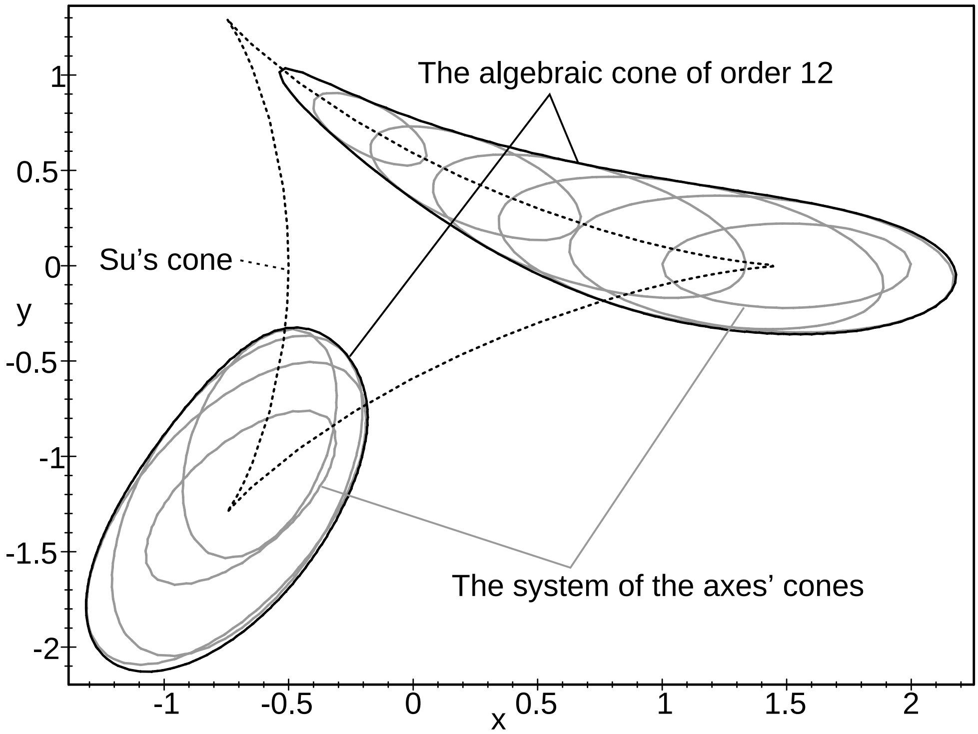


Figure 3: The system of the axes' cones (in grey), its envelope (black), and the cone of B. Su (dotted) in an elliptic surface point.

### 3.3 The polars of the affine normal with respect to the axes' cones

Consider again the axes' cone (24) of a tangent $T$. In this section the affine normal of the given surface is assumed to be defined using formula (16), i.e., we have either an elliptic or a hyperbolic surface point $\mathbf{O}$. Then, the polar plane of the affine normal $(\,\mu\, a^* \;\; \mu\, b^* \;\; \mu\,)^\top$ $(\mu \in \mathbb{R})$ is given by the equation

$$0 \;=\; (\,x \;\; y \;\; z\,) \cdot K(\xi,\eta) \cdot \begin{pmatrix} a^* \\ b^* \\ 1 \end{pmatrix} = q_1(\xi,\eta) \cdot x + q_2(\xi,\eta) \cdot y + q_3(\xi,\eta) \cdot z\,.$$

Resulting from (6), the three functions $q_1(\xi,\eta), q_2(\xi,\eta)$ and $q_3(\xi,\eta)$ are quartic polynomials in $\xi$ and $\eta$. Corresponding to the system of axes' cones of all tangents at the surface point $\mathbf{O}$ we get a system of polars (30) of the affine normal.

**Proposition 7.** *In the general case, the polar planes (30) of the affine normal (16) with respect to the system of all axes cones (24) envelop a rational cone of algebraic class 4 and order 6 through the (elliptic or hyperbolic) point* $\mathbf{O}$*. The intersection of each polar (30) with the tangential plane is a tangent* $G$ *through* $\mathbf{O}$*, which depends only on the local properties of third order (i.e., on the partial derivatives of maximal order 3) of the given surface.*

*Proof.* The second part of the proposition remains to be shown. It results immediately from the fact, that the coefficients of the quartic polynomials $q_1(\xi, \eta)$ and $q_2(\xi, \eta)$ (which can be computed from (5) and (30)) depend only on the coefficients $f_{i,j}$ of the expansion (1) with $i + j \leq 3$. □

As a second example, Figure 4 shows the system of the axes' cones, the rational cone of class 4 enveloped by the polars of the affine normal, and the cone of B. Su for a surface with a hyperbolic point **O**. Again, the intersections of these cones with the plane $z = 1$ have been drawn.

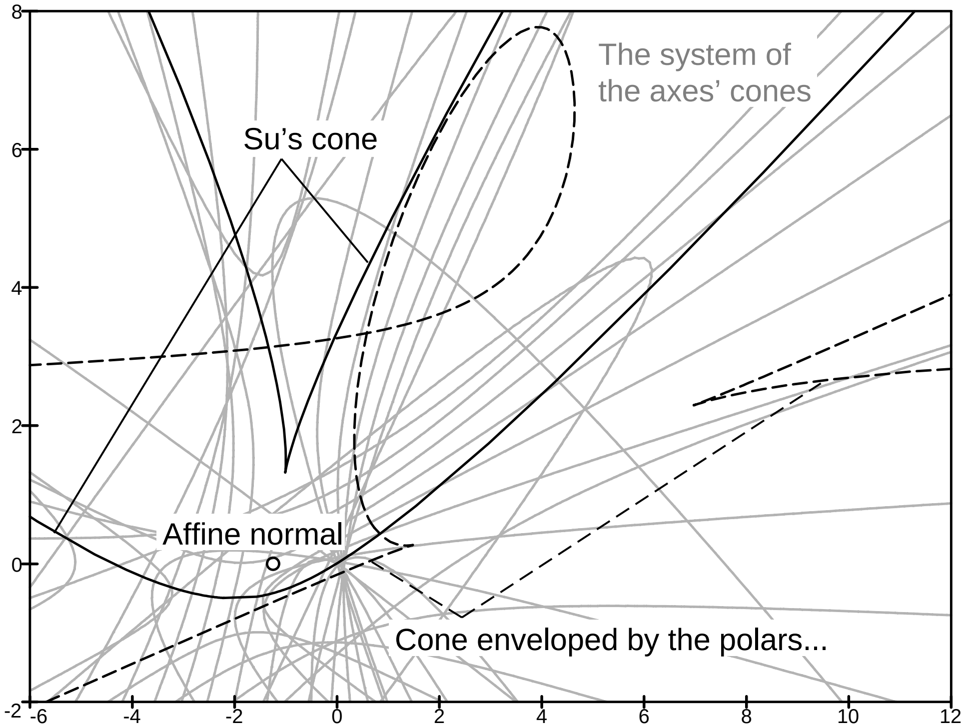


Figure 4: The system of the axes' cones (in grey), the envelope of the polars of the affine normal (dashed), and the cone of B. Su (solid) in a hyperbolic surface point.

On the one hand, the rational cone from this proposition characterizes the distribution of the axes' cones (24). Hence, it can be used as a visualization of the local surface properties. On the other hand, there is no direct relationship between the cubic paraboloids $P_3^{(a,b)}(\xi, \eta)$ with multiple osculating tangents and the axes directions which are contained in the rational cone. Especially, the cuspidal edges and multiple lines of the cone have no direct meaning for the cubic osculating paraboloids. Therefore the discussion of the algebraic cone of order 12 introduced in the previous section seems to be more promising, but it is also decisively more complicated.

## 4. Flat points

For the sake of completeness we briefly discuss the case of flat surface points which has been excluded from the previous considerations. These surface points are characterized by $\varphi_2(x,y) \equiv 0$, i.e., the tangential plane $z = 0$ and the surface have a contact of (at least) third order at the point $\mathbf{O}$. Under these assumptions, the terms of the Taylor expansion (3) with respect to the affine coordinate system (2) degenerate to

$$\begin{aligned} &\psi_2^{(a,b)}(\xi,\eta) \equiv 0 \\ &\psi_3^{(a,b)}(\xi,\eta) = \varphi_3(\xi,\eta) \\ &\psi_4^{(a,b)}(\xi,\eta) = \varphi_4(\xi,\eta) \quad . \end{aligned} \tag{30}$$

Thus, all osculating paraboloids (9) of second order degenerate to the tangential plane of the surface. The three osculating tangents (i.e. the tangents of the intersection with the surface at $\mathbf{O}$) of the tangential plane are the three null directions of the cubic form $\varphi_3(x,y)$, provided that $\varphi_3(x,y) \not\equiv 0$ holds (the tangential plane is presumed to have no contact of higher than second order with the surface). The osculating tangents may be multiple or two of them may be conjugated–complex.
The cubic osculating paraboloids result from (21) and (30). Due to $\psi_4^{(a,b)}(\xi,\eta) = \varphi_4(\xi,\eta)$, all these cubic paraboloids possess the same osculating tangents, provided that $\varphi_4(x,y) \not\equiv 0$ holds (they are presumed to have no contact of higher than third order with the given surface). Again, the osculating tangents may be multiple, or two of them may be conjugated–complex.
In the neighbourhood of flat surface points, the null directions of the cubic forms $\varphi_3(x,y)$ and $\varphi_4(x,y)$ yield an affinely invariant characterization of the surface properties depending on partial derivatives of maximal order four. In contrast with this, these null directions are not affinely invariant for general non–flat surface points.

## Final remark

The discussion of osculating paraboloids gave rise to the definition of certain cones associated with each surface point. These cones characterize the distribution of the Transon planes or of the axes' cones at the given surface point. So we can derive an affinely invariant visualization of the local surface properties: Su's cone and the algebraic cone of Proposition 6 characterize the properties which depend on derivatives of maximal order 3 and 4, respectively. Another visualization of the third order properties of a surface is given by the cubic indicatrix (see [8, 9]), but this curve is not affinely invariant.
As a consequence, we get a geometric interpretation of the order of contact between two surfaces. This notion has a long history in differential geometry, cf. [12]. During the last years, it has become a subject of renewed interest as "Geometric Continuity" in Computer Aided Geometric Design, see [7]. Usually, the contact of $n$–th order between two given

surface is defined by the existence of a reparameterization ensuring a $C^n$–joint of the two parametric representations. For a contact of order $n \leq 2$, the geometric characterization is obvious: first and second order contacts mean common tangential planes and (additional) coinciding Dupin indicatrices, respectively. Now we are able to derive a similar affine invariant characterization for contacts of third and fourth order: Two surfaces have at a common point a contact of order three, if the tangential planes, the Dupin indicatrices, and the cones of B. Su are identical. Moreover, they possess a fourth–order–contact, if for each tangent the corresponding generators of the rational cone which is enveloped by the polars of the affine normal with respect to the axes' cones (see Proposition 7) coincide.

### Appendix 1: The proof of Lemma 1

This Maple program generates the coefficients of the Taylor expansion (3).

```
> f := array(0..4, 0..4):
> phi := array(2..4):
> # Taylor expansion of the given surface in xyz coordinates:
> for i from 2 to 4 do
>  phi[i] := sum( 'f[j,i-j] * x^j * y^(i-j)', 'j'=0..i);
> od;
> phisum := phi[2] + phi[3] + phi[4];
> g := array(0..4, 0..4):
> psi := array(2..4):
> # Taylor expansion with respect to affine coordinates:
> for i from 2 to 4 do
>  psi[i] := sum( 'g[j,i-j] * xi^j * eta^(i-j)', 'j'=0..i);
> od;
> psisum := psi[2] + psi[3] + psi[4];
> # The Parametric representation of psisum in xyz coordinates...
> X := xi + a * psisum;
> Y := eta + b * psisum;
> Z := psisum;
> # is inserted into the implicit equation of the given surface:
> t := collect( subs(x=X, y=Y, z=Z, seq( xi^i=0, i=5..16),
>  seq(eta^i=0, i=5..16), expand(z-phisum)), [xi,eta],distributed );
> eqn := array(0..4, 0..4):
> # All terms of total degree < 5 in xi,eta must vanish:
> for i from 0 to 4 do
>  for j from 0 to 4-i do
>    eqn[i,j] := coeff(coeff(t, xi, i), eta, j)=0;
>  od;
> od;
> # Computation of the coefficients g[i,j]
> eqs := seq(seq(eqn[i,j], j=0..4-i), i=0..4):
> vars := seq(seq(g[i,j], j=0..4-i), i=0..4):
> assign(solve(eqs, vars));
> # The result:
> seq(print('i'=i, ' psi[i]'=sort(psi[i], [a,b])), i=2..4);
```

## Appendix 2: The proof of Proposition 6

This Maple program can be used after performing the calculations of Appendix 1. We use the canonic equations of elliptic or hyperbolic surface points as derived in [2, p. 109/110] because otherwise the formulae would become too large.

```
> # canonical equation of an hyperbolic point - type 1
>  f[2,0]:=1/2:  f[1,1]:=0:  f[0,2]:=-1/2:
>  f[3,0]:=C/6:  f[2,1]:=0:  f[1,2]:=1/2*C: f[0,3]:=0:
> # (canonical equation of an hyperbolic point - type 2
> # f[2,0]:=1/2:  f[1,1]:=0:  f[0,2]:=-1/2:
> # f[3,0]:=1/6:  f[2,1]:=1/2:  f[1,2]:=1/2:  f[0,3]:=1/6:)
> # (canonical equation of an elliptic point
> # f[2,0]:=1/2:  f[1,1]:=0:  f[0,2]:=1/2:
> # f[3,0]:=C/6:  f[2,1]:=0:  f[1,2]:=-1/2*C: f[0,3]:=0:)
> with(linalg):
> # generating the 3x3-matrix of the axes' cone
> rkmat := matrix(3,3,[
> sort(coeff(expand(psi[4]),a^2),[xi,eta]),
>        1/2*sort(coeff(coeff(expand(psi[4]),a),b),[xi,eta]),
>            1/2*sort(coeff(expand(subs(b=0, psi[4])),a),[xi,eta]),
>      1/2*sort(coeff(coeff(expand(psi[4]),a),b),[xi,eta]),
>        sort(coeff(expand(psi[4]),b^2),[xi,eta]),
>            1/2*sort(coeff(expand(subs(a=0, psi[4])),b),[xi,eta]),
>      1/2*sort(coeff(expand(subs(b=0, psi[4])),a),[xi,eta]),
>        1/2*sort(coeff(expand(subs(a=0, psi[4])),b),[xi,eta]),
>            sort(subs(a=0,b=0,expand(psi[4])),[xi,eta])  ] ):
> # equation of the axes' cone in xyz coordinates
> psi4xyz := multiply( transpose(vector(3,[x,y,z])), rkmat, vector(3,[x,y,z]));
> # the two equations of the cone enveloped by the axes' cones
> # yields two cubic polynomials
> collect(collect(psi[4],eta),xi);
> xyzpoly1 := collect(collect( diff(psi4xyz,eta)  ,eta),xi);
> xyzpoly2 := collect(collect( 4/xi*psi4xyz - eta/xi*diff(psi4xyz,eta), eta),xi);
> # the coefficients of these polynomials -> two matrix rows
> matrow1 := seq(coeff(coeff(xyzpoly1, xi,i),eta,3-i), i=0..3);
> matrow2 := seq(coeff(coeff(xyzpoly2, xi,i),eta,3-i), i=0..3);
> # dependencies between the xi^i*eta^(3-i) -> three matrix rows
> matrow3 := 1,-lambda,0,0;
> matrow4 := 0,1,-lambda,0;
> matrow5 := 0,0,1,-lambda;
> # from the five matrix rows we construct three 4x4-matrices
> # their determinants are three quadratic polynomials in lambda
> resupol1 := det(matrix(4,4,[matrow1,matrow2,matrow3,matrow4]));
> resupol2 := det(matrix(4,4,[matrow1,matrow2,matrow3,matrow5]));
> resupol3 := det(matrix(4,4,[matrow1,matrow2,matrow4,matrow5]));
> # these three polynomials must have a common root, their
> # resultant yields the algebraic equation of the enveloped cone
> implequ := collect(expand(
>    det(matrix(3,3,[seq(coeff(resupol1,lambda,i),i=0..2),
>                    seq(coeff(resupol2,lambda,i),i=0..2),
>                    seq(coeff(resupol3,lambda,i),i=0..2) ]))),[x,y,z]):
```

```
> # the order of the algebraic cone is equal to:
> degree(implequ,[x,y,z]);
                                      12
> # generally it cannot be factorized
> factor(implglequ);
                                     (...)
> # intersection with the tangential plane:  asymptotic tangents
> factor(subs(z=0,implequ));
                                        6         6
                             5/4 (x - y)  (x + y)
```

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