Spline Implicitization of Planar Curves

Bert Jüttler, Josef Schicho, and Mohamed Shalaby

Abstract. We present a new method for constructing a low degree implicit spline representation of a given parametric planar curve. To ensure the low degree condition, quadratic B-splines are used to approximate the given curve via orthogonal projection in Sobolev spaces. Adaptive knot removal, which is based on spline wavelets, is used to reduce the number of segments. The B-spline segments are implicitized, and the resulting bivariate functions are joined along suitable transversal lines, yielding a globally continuous bivariate function. As shown by analyzing the asymptotic behavior of these transversal lines for step size $h \to 0$, the given curve can be implicitized with any desired accuracy.

§1. Introduction

Planar curves in Computer Aided Geometric Design can be defined in two different ways. In most applications, they are described by a *parametric representation*, $x = x(t)/w(t)$ and $y = y(t)/w(t)$, where $x(t), y(t)$ and $w(t)$ are often polynomials or piecewise polynomials. Alternatively, the *implicit form* $f(x, y) = 0$ can be used. Both the parametric and implicit representation have their advantages. The availability of both often results in simpler and more efficient computations. For example, if both representations are available, the intersection of two curves can be found by solving a one–dimensional root finding problem.

Any rational parametric curve has an implicit representation, while the converse is not true. The process of converting the parametric equation into implicit form is called implicitization. A number of established methods for *exact* implicitization exists: resultants, Gröbner bases, and moving curve and surface, see e.g. [3] for further information. However, *exact* implicitization has not found widespread use in CAGD. This is – among other reasons – due to the following facts:

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- Exact implicitization often produces large data volumes, as the resulting implicit polynomials may have a huge number of coefficients.
- The exact implicitization process is relatively complicated, especially in the case of high polynomial degree. For instance, most resultant–based methods need the symbolic evaluation of large determinants.
- Even for regular parametric curves, the exact implicitization may have unwanted branches or self–intersections in the region of interest.

For these reasons, approximate implicitization has been proposed. Several methods are available: Montaudouin and Tiller [10] use power series to obtain local explicit approximation (about a regular point) to polynomial parametric curves and surfaces. Chuang and Hoffmann [2] extend this method using what they called "implicit approximation". Dokken [4] proposes a new way to approximate the parametric curve or surface globally; the approximation is valid within the whole domain of the curve segment or surface patch. Sederberg et al. [12] use monoid curves and surfaces to find an approximate implicit equation and approximate inversion map of a planar rational parametric curve or a rational parametric surface .

This paper presents a method for constructing what we call a spline implicitization for planar curves: a partition of the plane into polygonal segments, and a bivariate polynomial for each segment, such that the collection of the zero contours approximately describes the given curve. The polynomial pieces form a globally C^m spline function for a certain choice of m. In this paper we restrict ourselves to continuous functions ($m = 0$). A method for C^1 spline implicitization is currently under investigation [7].

§2. B–spline Approximation of Planar Curves

Following the technique proposed in [11], we generate a quadratic B–spline approximation via orthogonal projection in Sobolev spaces. The quadratic B–splines B_j on $[0,1]$ with uniform knots (step size $h = 2^{-i}$) and 3-fold boundary knots form an orthonormal sequence in a suitably weighted Sobolev space. In the interior of the segment, the inner product is defined as

$$\langle f, g\rangle = \frac{1}{h}\,(f,g) + \frac{1}{4}h\,(f',g') + \frac{1}{30}h^3\,(f'',g''), \tag{1}$$

where $(.,.)$ is the usual L^2 inner product. In order to achieve orthogonality at the boundary segments, additional terms have to be used. These weights and weight matrices (which are used near the boundary), have been specified in [11].

The B-spline approximation $\mathbf{g}^*$ of a given curve $\mathbf{g}(t) = (g_1(t), g_2(t))$ with respect to the norm which is induced by the inner product (1), can

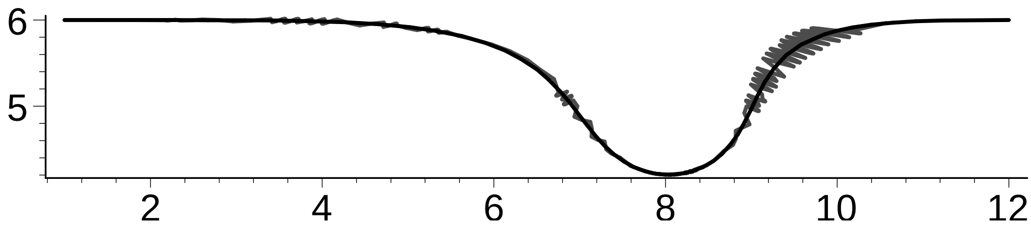

Fig. 1. The original curve (black) and the error introduced by approximating it with a quadratic B–spline curve (gray, exaggerated by a factor of 25,000).

then be written as

$$\mathbf{g}^*(t) = \sum_j \mathbf{d}_j \, B_j(t) \quad \text{with} \quad \mathbf{d}_j = \begin{pmatrix} \langle g_1, B_j \rangle \\ \langle g_2, B_j \rangle \end{pmatrix}. \tag{2}$$

The control points $\mathbf{d}_j$ of the approximating B–spline curve can be generated by simple and efficient computations, as only (possibly numerical) integrations are needed. Also, no assumption about the given parametric representation have to be made, except that it should be at least in the underlying Sobolev space $H^{2,2}$. By using sufficiently many segments, an arbitrarily accurate approximation can be generated; the approximation order is 3.

Example. Consider the polynomial parametric curve $\mathbf{g}$ of degree 20 which is shown in Fig. 1 (black curve). First, we approximate $\mathbf{g}$ using quadratic B–splines. Fig. 1 shows the error between $\mathbf{g}$ (black) and the quadratic B–spline approximation $\mathbf{g}^*$ (gray) for step size $h = 1/128$. Note that the error had to be exaggerated by a factor $\delta = 25,000$ to make it visible.

§3. Data Reduction via Spline Wavelets

After computing the initial B–spline approximation, we apply a knot removal procedure in order to reduce the number of segments. Knot removal has been discussed in a number of publications, see e.g. [5,9] and the references cited therein. In the present work, we use a simple and efficient method which is adapted to our special situation. It is based on spline wavelets [1]. The method is not optimal, but it is cheaper than all other methods, since no sorting or ranking lists (as used in [5]) are required.

First we compute the wavelet transform of the initial B-spline approximation. Then, by setting all wavelets coefficients vectors with norm less than a certain threshold to zero, we remove blocks of wavelets. For each block, one of the two common knots can be removed from the knot sequence. The length of these blocks varies between 2 (for knots close to the boundary) and 5 (for inner knots) wavelet coefficents. Finally, we generate the optimized approximative B–spline representation $\mathbf{g}^{**}$ of the given curve over the reduced knot sequence.

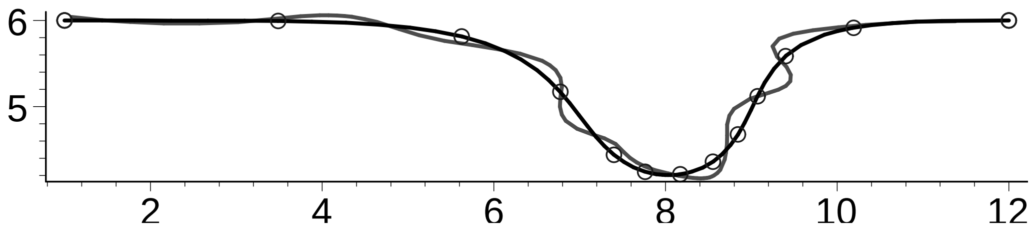

Fig. 2. The original curve (black) and error introduced by the B–spline approximation after the data reduction (gray, exaggerated by a factor of 10).

Error bounds can be generated by applying the wavelet synthesis to the set of removed wavelets. Due to the convex hull property of the B–splines, the error is bounded by the maximum absolute value of the resulting Bézier control points.

Example (continued). We apply this procedure to the quadratic B–spline curve $\mathbf{g}^*$. The number of knots is reduced from 133 to 17, where the threshold is equal to 10^{-4}. Fig. 2 shows the error between the original curve $\mathbf{g}$ (black) and the final B–spline representation curve $\mathbf{g}^{**}$ (gray) over K_{final}. The knots are shown as circles. The error is exaggerated by a factor $\delta = 10$ to make it visible.

§4. Segmentwise Implicitization

After the data reduction process, we obtain a quadratic B–spline approximation $\mathbf{g}^{**}$ defined over a nonuniform knot sequence. In order to implicitize this curve, we split the B–spline representation of this curve into Bézier segments, using knot insertion. Then, each quadratic Bézier segment is implicitized.

Each quadratic parametric Bézier segment has three control points. Let (p_0, q_0), (p_1, q_1) and (p_2, q_2) be the control points of one of these segments. The implicit form of this segment can be shown to be

$$G(x, y) = \det \begin{pmatrix} Q_0(y)P_2(x) - P_0(x)Q_2(y) & Q_0(y)P_1(x) - P_0(x)Q_1(y) \\ Q_1(y)P_2(x) - P_1(x)Q_2(y) & Q_0(y)P_2(x) - P_0(x)Q_2(y) \end{pmatrix}$$

where $P_i(x) = \binom{2}{i}(p_i - x)$ and $Q_i(y) = \binom{2}{i}(q_i - y)$, for $i = 0, \ldots, 2$. This results in a sequence of quadratic bivariate polynomials, one for each Bézier segment of the spline curve $\mathbf{g}^{**}$.

§5. Joining the Segments

In order to generate a continuous function, we join the bivariate polynomials which have been produced by the implicitization process along suitable transversal lines.

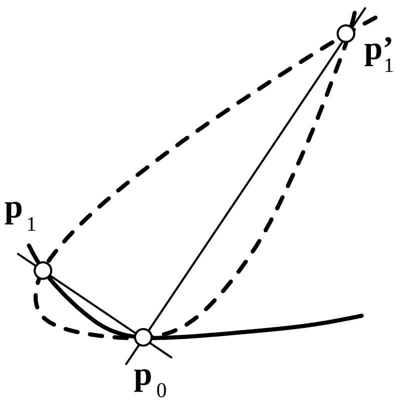

Fig. 3. Choosing the transversal line.

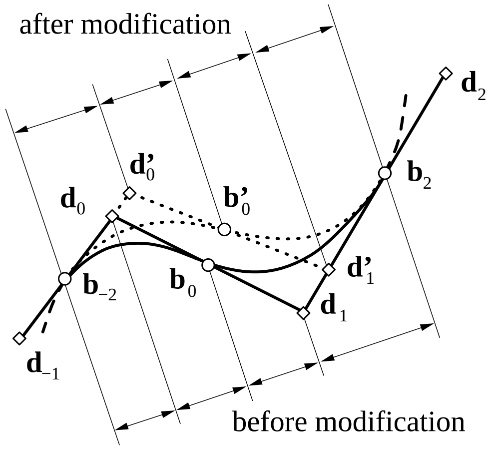

Fig. 4. Modification of the control points at an inflection, see Section 5.3.

5.1 Defining a continuous function

Consider two neighboring Bézier segments of $\mathbf{g}^{**}$ with implicit representations $G_i(x, y)$, $i = 1, 2$. These segments are parabolas which meet with tangent continuity at their junction point $\mathbf{p}_0$. Moreover, they intersect in two additional points $\mathbf{p}_1, \mathbf{p}'_1$, see Fig. 3. These two points can be real, conjugate complex, or even at infinity. Obviously, the transversal line has to be chosen as the line passing through the junction point $\mathbf{p}_0$ and one of these other two intersection points $\mathbf{p}_1$, $\mathbf{p}'_1$. If these points are real, then there are two possibilities to choose this line. We pick the line $L(\mathbf{p}_0, \mathbf{p}_1)$ which is closer to the normal vector of the curve. According to the following lemma, we can then always achieve a C^0 joint along the transversal line $L(\mathbf{p}_0, \mathbf{p}_1)$.

Lemma 1. *Suppose we are given two quadratic functions $G_1(x, y)$ and $G_2(x, y)$ such that they have a common root and a parallel gradient at $\mathbf{p}_0$, and intersect at $\mathbf{p}_1$. Let $L(\mathbf{p}_0, \mathbf{p}_1)$ be the line joining $\mathbf{p}_0$ and $\mathbf{p}_1$. Then after multiplying G_2 by a suitable constant, $G_1(x, y)$ and $G_2(x, y)$ are identical along the line $L(\mathbf{p}_0, \mathbf{p}_1)$.*

Proof: The restrictions of $G_1(x, y)$ and $G_2(x, y)$ to the line $L(\mathbf{p}_0, \mathbf{p}_1)$ are two quadratic functions with common roots at $\mathbf{p}_0$ and $\mathbf{p}_1$. After multiplying G_2 by a suitable constant, they are identical along this line. □

After multiplying the implicitized segments with suitable constants, the collection of these bivariate polynomials – each restricted to a tile bounded by the transversal lines as described in Lemma 1 – forms a continuous function. This function is defined within a certain neighborhood of the curve. (Clearly, it can be extended continuously beyond this neighborhood, using additional polynomial segments.)

5.2 Asymptotic Behavior of the Transversal Lines

We investigate the behavior of the transversal lines for decreasing step size (= segment length) h. For the sake of simplicity, we assume that the original curve is given by an arc length parameterization. It is well known that no polynomial and rational curves – except straight lines – can be equipped with a closed–form arc length parameterization [6]. However, one can always reparameterize a general parametric curve by its arc length approximately, using numerical methods.

Theorem 2. *Consider a C^3 curve which is parameterized by its arc length, and which has no inflection point, $\kappa \neq 0$. We apply the process of spline implicitization (orthogonal projection in Sobolev space, implicitization, and joining of the segments) to the curve, where the knots of the quadratic B-Spline curve are uniformly spaced with step size h. If the step size h tends to zero, then the transversal line $L(\mathbf{p}_0, \mathbf{p}_1)$ gets closer and closer to the normal vector of the curve at the point $\mathbf{p}_0$.*

Proof: Our analysis is based on the so–called canonical Taylor expansion [8] of the curve, which is derived from the Frenet–Serret formulas in elementary differential geometry. This expansion is given by

$$\mathbf{p}(s) = \begin{pmatrix} s - \frac{1}{6}\,\kappa_0^2\, s^3 - \frac{1}{8}\,\kappa_0\,\kappa_1\, s^4 + \mathcal{O}(s^5) \\ \frac{1}{2}\,\kappa_0\, s^2 + \frac{1}{6}\,\kappa_1\, s^3 + \frac{1}{24}\,(\kappa_2 - \kappa_0^3)\, s^4 + \mathcal{O}(s^5) \end{pmatrix}, \tag{3}$$

with $\kappa_0 = \kappa(0)$, $\kappa_1 = (d/ds)\,\kappa(0)$, etc. First, we approximate the curve $\mathbf{p}(s)$ with a quadratic B–spline curve defined over a uniform knot sequence $(..., -2h, -h, 0, h, 2h, ...)$, as described in Section 2. We consider the two neighboring Bézier segments $\mathbf{p}_{\text{left}}$ and $\mathbf{p}_{\text{right}}$ with the parameter domains $[-h, 0]$ and $[0, h]$ respectively. Using the expansion (3), we generate Taylor expansions for the B-spline control points $\mathbf{d}_{-1}, \mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2$ of both segments,

$$\mathbf{d}_{-1|2} = \begin{pmatrix} \mp\frac{3}{2}h \pm \frac{3}{8}\kappa_0^2 h^3 + \mathcal{O}(h^4) \\ \kappa_0 h^2 \mp \frac{3}{8}\kappa_1 h^3 + \mathcal{O}(h^4) \end{pmatrix}, \mathbf{d}_{0|1} = \begin{pmatrix} \mp\frac{1}{2}h \mp \frac{1}{24}\kappa_0^2 h^3 + \mathcal{O}(h^4) \\ \pm\frac{1}{24}\kappa_1 h^3 + \mathcal{O}(h^4) \end{pmatrix},$$

where $\mathbf{d}_i$ is associated with the knots $(i-1)h$ and ih. In order to find the intersection points $\mathbf{p}_1, \mathbf{p}_1'$, we implicitize $\mathbf{p}_{\text{left}}$, and substitute the parametric form of $\mathbf{p}_{\text{right}}$ into the implicit form of $\mathbf{p}_{\text{left}}$. This gives a quartic equation in the curve parameter S of the right segment, where $s = hS$. The factor S^2 factors out, as both parabolic arcs are joined with tangent continuity. Solving the remaining quadratic equation we get two values S_1 and S_2 of the parameter S. By substituting these values into $\mathbf{p}_{\text{right}}$, we obtain two Laurent series for the additional intersections,

$$\mathbf{p}_1 = \begin{pmatrix} -\frac{\kappa_1}{\kappa_0^3} + \frac{2}{3}\frac{(3\kappa_1^2 - \kappa_0\kappa_2 + \kappa_0^4)}{\kappa_0^4} h + \mathcal{O}(h^2) \\ \frac{8}{\kappa_0^3} h^{-2} - \frac{1}{6}\frac{(39\kappa_1^2 + 12\kappa_0^4 - 20\kappa_0\kappa_2)}{\kappa_0^5} + \mathcal{O}(h) \end{pmatrix} \tag{4}$$

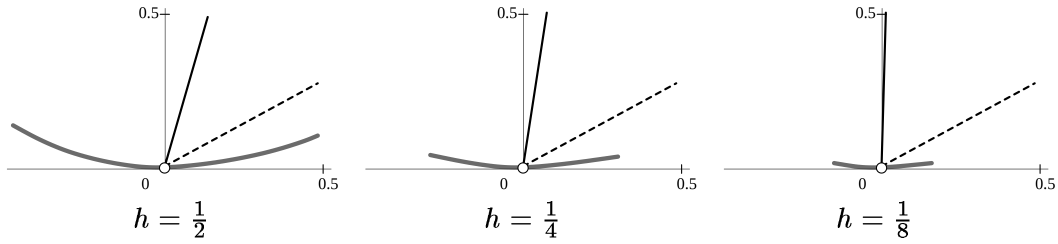

Fig. 5. The transversal lines for different step sizes.

and

$$\mathbf{p}_1' = \begin{pmatrix} -\dfrac{\kappa_1}{\kappa_0^3} - \dfrac{1}{12}\dfrac{\kappa_1(3\kappa_1^2 - \kappa_0\kappa_2 + 2\kappa_0^4)}{\kappa_0^5}h^2 + \mathcal{O}(h^3) \\ \dfrac{1}{2}\dfrac{\kappa_1^2}{\kappa_0^5} + \dfrac{1}{96}\dfrac{\kappa_1^2(15\kappa_1^2 - 4\kappa_0\kappa_2 + 12\kappa_0^4)}{\kappa_0^7}h^2 + \mathcal{O}(h^3) \end{pmatrix}. \tag{5}$$

Clearly, if the curve has no inflection at $\mathbf{p}_0$ (i.e., $\kappa_0 \neq 0$), then the first intersection point $\mathbf{p}_1$ converges for decreasing step size, $h \to 0$, to the infinite point of the normal at $\mathbf{p}_0$. The second intersection tends to a fixed limit position, which is fully determined by the curvature and its derivative with respect to arc length at $\mathbf{p}_0$. □

We illustrate this result by an example (see Fig. 5), showing the curve segments (gray) and the possible transversal lines (black lines, solid and dashed) for three step sizes. The black transversal line converges to the curve normal.

According to this result, the system of transversal lines behaves nicely, provided that the curve has no inflections, is parameterized by an arc length parameter, and is approximated with sufficiently small step size (without knot removal). Under these assumptions, the two additional intersections of the parabolas are real, and one of them can be brought arbitrarily close to the normal of the curve. In the limit, for step size (segment length) $h \to 0$, these transversal lines envelop the *evolute* of the given curve, see [8]. Thus, for sufficiently small step size h, we get a system of transversal lines through the junction points of neighboring segments, such that the implicit equations can be joined (at least) continuously along them.

According to our numerical experiments, the arc length parameterization is not really needed in practice, and also knot removal can be used, in order to reduce the number of segments. However, inflections of the given curve cause a problem which has to be dealt with.

5.3 Singular Case: Inflection Points

The above construction fails at inflections of the quadratic spline curve, as the additional intersection points $\mathbf{p}_1$, $\mathbf{p}_1'$ do not exist in this case. In this situation, the two implicitized segments can still be joined continuously, provided that they can be seen as a graph of a univariate quadratic spline *function* with respect to a suitable coordinate system; the parallels of the y-axis then form a system of suitable transversal lines. That is, the axes of both parabolas have to be parallel; the two parabolas then share the infinite point of the y-axis.

Generally, however, the axes of the two parabolic segments are not parallel. As an example, Fig. 4 (left) shows an inflected curve consisting of two parabolic segments G_1, G_2 and the control polygon. The B-spline control points $(\mathbf{d}_{-1}, \ldots, \mathbf{d}_2)$ are shown as diamonds, and the additional Bézier control points $(\mathbf{b}_{-2}, \ldots, \mathbf{b}_2)$ are shown as circles.

The axes of the parabolas are parallel if and only if there exists an auxiliary line G, such that the orthogonal projections of $\mathbf{d}_0$ and $\mathbf{d}_1$ are the midpoints of the projected line segments $\overline{\mathbf{b}_{-2}\mathbf{b}_0}$ and $\overline{\mathbf{b}_0\mathbf{b}_2}$. We choose the line G as a parallel to $\overline{\mathbf{b}_{-2}\mathbf{b}_2}$. In order to obtain parabolas with parallel axes, we adjust the location of the B-spline control points $\mathbf{d}_0$ and $\mathbf{d}_1$ on the lines $\overline{\mathbf{d}_{-1}\mathbf{d}_0}$ $\overline{\mathbf{d}_1\mathbf{d}_2}$, leading to modified points $\mathbf{d}_0'$ and $\mathbf{d}_1'$. The new locations can be found by a short calculation. Clearly, this process will introduce a minor error.

Example (finished). Using these techniques we derived a piecewise quadratic function G, whose zero contour $G(x, y) = 0$ is the quadratic B-Spline curve (see Fig. 2). In order to visualize the quality of the implicitization, Fig. 6 shows the level curves or algebraic offsets (thin lines), $G(x, y) = c$ for certain constants (algebraic distances) c, and the transversal lines (which define the polynomial pieces of G) through the junction points of the segments. In order to make the picture clearer, we enlarged a part of the curve. Generally, the algebraic offsets (which consist of parabolic arcs) are not tangent continuous.

§6. Conclusion

We have presented a method for generating an approximate implicit spline representation of a parametric planar curve which is valid within a certain neighborhood of the given curve. The construction consists of four steps: B–spline curve approximation, knot removal, segment implicitization, and segment joining. According to our numerical experience, the numerical reparameterization to arc length, although essential for theoretical analysis, is not needed in practice.

Compared to the existing methods for implicitization, our method has the following advantages.

- The method is computationally simple. In particular, no evaluations (symbolic or numerical) of large determinants are needed.

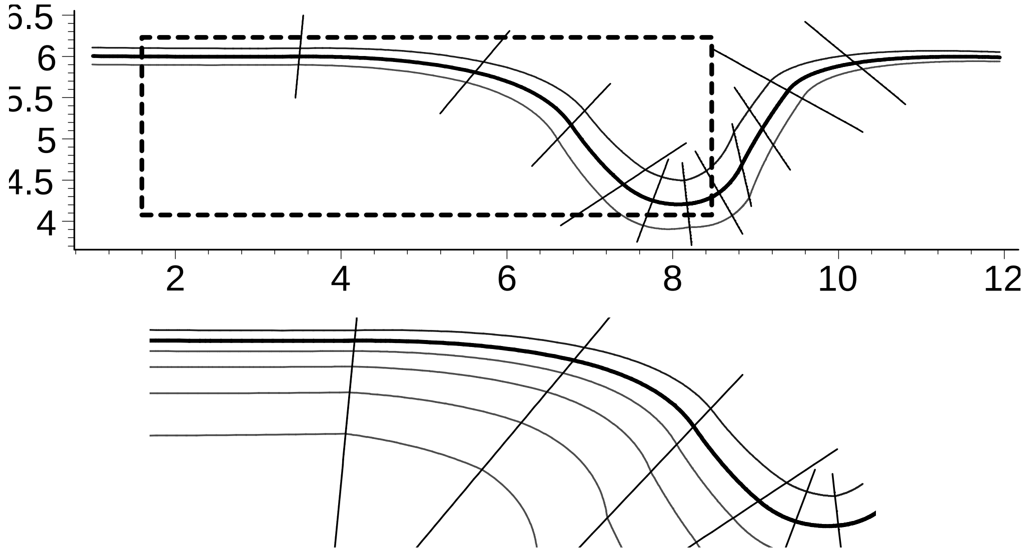

Fig. 6. Implicitized curve and algebraic offsets.

- It produces a low degree implicit representation. For instance, the intersection of a line with the implicitized curve can be found by computing the roots of a quadratic polynomial.
- Due to the low degree, the methods avoids unwanted branches or singularities, which otherwise could be present in the neighborhood of the given curve.
- The implicit function is globally continuous.
- The method can be applied to any parametric curve with coordinate functions in the Sobolev space $H^{2,2}$, not just to polynomial or piecewise polynomial representations.
- As compared to the exact implicitization, the method yields – for high degree curves – a smaller data volume. In our example (degree 20), we have only 72 coefficients. Exact implicitization would produce 231 coefficients. Clearly, degrees as high as 20 are not realistic for applications. This advantage of the approximative implicitization, however, becomes more important in the surface case, where already bicubic patches give exact implicit representations involving 1330 coefficients.

As a matter of future research, we plan to generalize this method to the C^1 case, and to surfaces.

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