The dual basis functions for the Bernstein polynomials

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Abstract. An explicit formula for the dual basis functions of the Bernstein basis is derived. The dual basis functions are expressed as linear combinations of Bernstein polynomials.

Keywords: dual (reciprocal) basis, Bernstein polynomials, dual functional.

Subject classification: 41A10, 65D17.

1. Introduction

The Bernstein basis is frequently used both in Computer Aided Geometric Design and in Approximation Theory, see [7] and [9]. Its properties have been studied in an enormous number of publications, see e.g. [1, 2, 6]; it is virtually impossible to give a complete list here.

Associated with the Bernstein basis, there are the corresponding dual (also called inverse or reciprocal) basis functions with respect to the usual inner product of the Hilbert space $L^2[0,1]$. A recurrence relation for the dual basis functions has been found by Ciesielski [3]. This recurrence expresses any dual basis function of degree n as a linear combination of dual basis functions of degree $n-1$ and of the n–th Legendre polynomial.

The generalized Ball basis is another basis of polynomials which has applications in Computer Aided Geometric Design. A recent article by Othman and Goldman [11] presents an explicit formula for the dual basis functions for the generalized Ball basis of odd degree.

In the sequel we derive an explicit formula for the dual basis functions of the Bernstein polynomials, where these functions are again represented as linear combinations of the Bernstein basis. A brief discussion of the relation to dual functionals and to least–squares approximation is given in the end.

2. Bernstein polynomials and their dual basis

In the sequel we denote by $\mathcal{P}^n$ the $n+1$–dimensional real linear space of all polynomials of maximal degree n in the variable t, i.e.,

$$\mathcal{P}^n = \text{span}\{1, t, t^2, \dots, t^n\}. \tag{1}$$

We consider the elements of this space as real–valued functions $[0,1] \to \mathbb{R}$. By introducing the inner product $(.,.)$,

$$\big(\, f(t), g(t) \,\big) = \int_0^1 f(t)\, g(t)\, \mathrm{d}t \quad \text{for } f(t), g(t) \in \mathcal{P}^n, \tag{2}$$

the linear space $\mathcal{P}^n$ becomes a $n+1$–dimensional Hilbert space. The norm $||f(t)||$ of a polynomial $f(t) \in \mathcal{P}^n$ is given by $||f(t)|| = \sqrt{(f(t), f(t))}$.

The linear space $\mathcal{P}^n$ can be spanned by various systems of basis functions. For instance, the basis $\{1, t, t^2, \dots, t^n\}$ is called the monomial basis of this space. Another important basis is formed by the Bernstein polynomials $\{B_0^n(t), B_1^n(t), \dots B_n^n(t)\}$ of degree n with

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad (i = 0, \dots, n). \tag{3}$$

Like any basis of the space $\mathcal{P}^n$, the Bernstein polynomials have a unique dual basis $\{D_0^n(t), D_1^n(t), \dots, D_n^n(t)\}$ (sometimes also called the inverse or reciprocal basis) which consists of the $n+1$ dual basis functions

$$D_i^n(t) = \sum_{j=0}^{n} c_{i,j}\, B_j^n(t) \quad (j = 0, \dots, n). \tag{4}$$

We represent the dual basis functions with respect to the Bernstein basis. The real coefficients $c_{i,j} \in \mathbb{R}$ are unknown yet $(i, j = 0, \dots, n)$. The dual basis functions must satisfy the relation of duality, that is

$$\big(\, D_i^n(t), B_j^n(t) \,\big) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \quad (i, j = 0, \dots, n). \tag{5}$$

In Theorem 3 we will present an explicit formula for the coefficients $c_{i,j}$. The following two lemmas prepare the computation of these coefficients.

Lemma 1. *The inner product of two Bernstein polynomials evaluates to*

$$\big(\, B_i^m(t), B_j^n(t) \,\big) = \frac{\binom{m}{i}\binom{n}{j}}{(m+n+1)\binom{m+n}{i+j}}. \tag{6}$$

The proof immediately results from the definition (3) of the Bernstein polynomials, see e.g. [6].

Lemma 2. *The identity* $X_{p,r} = \delta_{p,r}$ *with*

$$X_{p,r} = \sum_{j=0}^{p} \sum_{q=j}^{n} (-1)^{p+q} \frac{(2j+1)\binom{n+j+1}{n-p}\binom{n-j}{n-p}\binom{n+j+1}{n-q}\binom{n-j}{n-q}}{(2n+1)\binom{2n}{q+r}} \tag{7}$$

holds for all integers p, r *with* $0 \le p, r \le n$.

Proof. By using the definition of binomial coefficients (see e.g. [8]) we get from (7)

$$X_{p,r} = \sum_{j=0}^{p} \frac{(-1)^{p+j}(2j+1)\binom{n+j+1}{n-p}\binom{n-j}{n-p}}{(n+j+1)\binom{2n+1}{n-j}\binom{n+j}{n-r}} \underbrace{\sum_{q=j}^{n} \binom{n+j+1}{n-q}\binom{-r-j-1}{q-j}\binom{2n-q-r}{n-r}}_{=Y_{r,j}}. \tag{8}$$

Substituting $q = n - k$ and applying the identity

$$\sum_{k=0}^{M-R+S} \binom{M-R+S}{k}\binom{N+R-S}{N-k}\binom{R+k}{M+N} = \binom{R}{M}\binom{S}{N} \tag{9}$$

from [8, Section 1.2.6, Exercise 31] with $R = n - r$, $M = j - r$, $N = n - j$ and $S = 2n + 1$ to $Y_{r,j}$ we obtain after some calculations

$$X_{p,r} = (-1)^{p+r} \frac{(n-r)!\,(n-r)!\,(2r)!}{(n-p)!\,(n-p)!\,(2p+1)!} \underbrace{\sum_{j=r}^{p} (2j+1)\binom{2p+1}{p-j}\binom{-2r-1}{j-r}}_{Z_{p,r}} \tag{10}$$

Note that $Y_{r,j}$ vanishes for $j < r$, therefore the sum in (10) runs from r to p only. Obviously we have $X_{p,r} = 0$ if $r > p$.

Assume $r < p$. Substituting $j = r + k$ and applying the identity

$$\sum_{k=0}^{M} \binom{R}{k}\binom{S}{M-k}(MR - (R+S)k) = 0 \tag{11}$$

from [8, Section 1.2.6, Exercise 53] with $R = -2r - 1$, $S = 2p + 1$ and $M = p - r$ we get $Z_{p,r} = 0$, hence $X_{p,r} = 0$ if $r < p$ holds. Finally, equation (10) yields $X_{p,p} = 1$. This proves the assertion. □

Now we are able to state the main result of this section:

Theorem 3. *The dual basis* $\{D_0^n(t), D_1^n(t), \ldots, D_n^n(t)\}$ *of the Bernstein basis* $\{B_0^n(t), B_1^n(t), \ldots, B_n^n(t)\}$ *of the Hilbert space* $\mathcal{P}^n$ *has the Bernstein–Bézier representation (4) with the real coefficients*

$$c_{p,q} = \frac{(-1)^{p+q}}{\binom{n}{p}\binom{n}{q}} \sum_{j=0}^{\min(p,q)} (2j+1)\binom{n+j+1}{n-p}\binom{n-j}{n-p}\binom{n+j+1}{n-q}\binom{n-j}{n-q} \qquad (12)$$

$(p, q = 0, \ldots n)$.

Proof. With the help of the Lemmas 1 and 2 we get

$$\begin{aligned}\left(D_p^n(t), B_r^n(t) \right) &= \sum_{q=0}^{n} c_{p,q} \left(B_q^n(t), B_r^n(t) \right) \\ &= \sum_{q=0}^{n} \frac{(-1)^{p+q}}{\binom{n}{p}\binom{n}{q}} \sum_{j=0}^{\min(p,q)} (2j+1)\binom{n+j+1}{n-p}\binom{n-j}{n-p}\binom{n+j+1}{n-q}\binom{n-j}{n-q} \frac{\binom{n}{q}\binom{n}{r}}{(2n+1)\binom{2n}{q+r}} \\ &= \frac{\binom{n}{r}}{\binom{n}{p}} X_{p,r} = \delta_{p,r} \ .\end{aligned} \qquad (13)$$

Note that the terms in the second row of (13) vanish if $q < j$ or $j > p$ hold, hence it is possible to apply Lemma 2. The polynomials $\{D_0^n(t), D_1^n(t), \ldots, D_n^n(t)\}$ obtained from (4) and (12) form the dual basis of the Bernstein polynomials because the conditions (5) of duality are satisfied. □

The Bernstein polynomials $\{B_0^n(t), \ldots, B_n^n(t)\}$ and the associated dual basis functions $\{D_0^n(t), \ldots, D_n^n(t)\}$ can be collected into $n+1$–dimensional column vectors

$$\mathbf{B}^n = \left(B_0^n(t)\ B_1^n(t) \ldots B_n^n(t) \right)^\top \quad \text{and} \quad \mathbf{D}^n = \left(D_0^n(t)\ D_1^n(t) \ldots D_n^n(t) \right)^\top . \qquad (14)$$

Resulting from Equation (4), these two vectors are related by the linear transformation $\mathbf{D}^n = C\,\mathbf{B}^n$ with the real $(n+1) \times (n+1)$–matrix $C = (c_{p,q})_{p,q=0,\ldots,n}$. The transformation matrix C is symmetric ($c_{p,q} = c_{q,p}$) and centra–symmetric ($c_{p,q} = c_{n-p,n-q}$). It is the inverse matrix of the matrix of the inner products $(B_i^n, B_j^n)_{i,j=0,\ldots,n}$ of the Bernstein polynomials (also called Gram matrix). Moreover, it can be decomposed into the product $C = L\,L^\top$ of the lower triangular matrix $L = (l_{i,j})_{i,j=0,\ldots,n}$ and its transpose $L^\top$. The components of L are

$$l_{i,j} = (-1)^{i+j} \frac{\binom{n+j+1}{n-i}\binom{n-j}{n-i}}{\binom{n}{i}} \sqrt{2j+1}\,. \qquad (15)$$

Note that $j > i$ implies $l_{i,j} = 0$.

As an example, Fig. 1 shows the quartic Bernstein polynomials $B_0^4(t), \ldots, B_4^4(t)$ and the corresponding dual basis functions $D_0^4(t), \ldots, D_4^4(t)$.

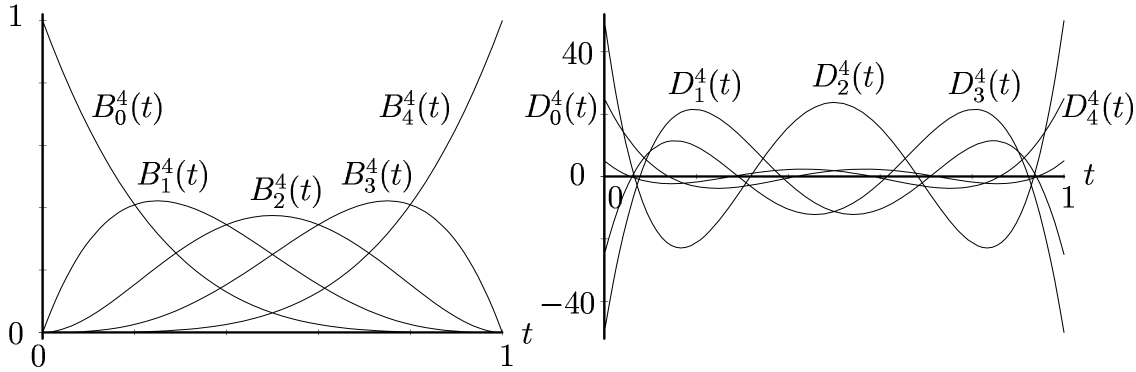

Figure 1: The Bernstein basis of degree 4 and the associated dual basis functions.

3. Polynomials satisfying boundary constraints

The formula for the dual basis functions can easily be generalized to linear spaces of polynomials with boundary conditions. Let

$$\mathcal{P}^{(n,k)} = \left\{ p(t) \in \mathcal{P}^n : \left.\frac{\mathrm{d}^i}{\mathrm{d}t^i}\, p(t)\right|_{t=0} = \left.\frac{\mathrm{d}^i}{\mathrm{d}t^i}\, p(t)\right|_{t=1} = 0 \quad \text{for} \quad i = 0, \ldots, k-1 \right\} \tag{16}$$

be the $(n+1-2k)$–dimensional Hilbert space of all polynomials with maximal degree n, whose 0th , ..., $k-1$st derivatives at $t=0$ and $t=1$ vanish ($k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$). The Bernstein polynomials $\{B_k^n(t), \ldots, B_{n-k}^n(t)\}$ form a basis of this space. For $k=0$ we set $\mathcal{P}^{(n,0)} = \mathcal{P}^n$.

Similarly to the previous section, the Bernstein basis $\{B_k^n(t), \ldots, B_{n-k}^n(t)\}$ of $\mathcal{P}^{(n,k)}$ has unique a dual basis $\{D_k^{(n,k)}(t), \ldots, D_{n-k}^{(n,k)}(t)\}$. The dual basis functions are represented as linear combinations of Bernstein polynomials,

$$D_j^{(n,k)}(t) = \sum_{i=k}^{n-k} c_{j,i}^{(k)}\, B_i^n(t) \quad j = k, \ldots, n-k \quad , \tag{17}$$

with certain real coefficients $c_{j,i}^{(k)}$. They must satisfy the relation

$$\left(D_i^{(n,k)}(t), B_j^n(t) \right) = \delta_{i,j} \quad (i, j = k, \ldots, n-k) \tag{18}$$

of duality. Obviously we have

$$D_j^{(n,0)}(t) = D_j^n(t), \quad \text{hence} \quad c_{j,i}^{(0)} = c_{j,i}\ , \tag{19}$$

see Theorem 3 ($i, j = 0, \ldots, n$). The remaining coefficients $c_{j,i}^{(k)}$ can be computed recursively:

Proposition 4. *The dual basis* $\{D_k^{(n,k)}, \ldots, D_{n-k}^{(n,k)}\}$ *of the Bernstein basis* $\{B_k^n(t), \ldots, B_{n-k}^n(t)\}$ *of the Hilbert space* $\mathcal{P}^{(n,k)}$ *has the Bernstein–Bézier representation (17), where the real coefficients* $c_{j,i}^{(k)}$ *satisfy the recurrence relation*

$$c_{j,i}^{(k)} = c_{j,i}^{(k-1)} - \alpha_j^{(k)} c_{k-1,i}^{(k-1)} - \beta_j^{(k)} c_{n-k+1,i}^{(k-1)} \quad (i = k, \ldots, n-k) \tag{20}$$

with

$$\alpha_j^{(k)} = \frac{c_{j,k-1}^{(k-1)} c_{n-k+1,n-k+1}^{(k-1)} - c_{j,n-k+1}^{(k-1)} c_{n-k+1,k-1}^{(k-1)}}{c_{k-1,k-1}^{(k-1)} c_{n-k+1,n-k+1}^{(k-1)} - c_{k-1,n-k+1}^{(k-1)} c_{n-k+1,k-1}^{(k-1)}} \quad \text{and}$$

$$\beta_j^{(k)} = \frac{c_{j,n-k+1}^{(k-1)} c_{k-1,k-1}^{(k-1)} - c_{j,k-1}^{(k-1)} c_{k-1,n-k+1}^{(k-1)}}{c_{k-1,k-1}^{(k-1)} c_{n-k+1,n-k+1}^{(k-1)} - c_{k-1,n-k+1}^{(k-1)} c_{n-k+1,k-1}^{(k-1)}} \tag{21}$$

$(j = k, \ldots, n-k;\ k = 1 \ldots \lfloor \frac{n}{2} \rfloor)$. *Hence, the dual basis functions fulfill*

$$D_j^{(n,k)}(t) = D_j^{(n,k-1)}(t) - \alpha_j^{(k)} D_{k-1}^{(n,k-1)}(t) - \beta_j^{(k)} D_{n-k+1}^{(n,k-1)}(t)\ . \tag{22}$$

Proof. The proof results by induction over k. We assume that the polynomials $D_{k-1}^{(n,k-1)}(t), \ldots, D_{n-k+1}^{(n,k-1)}(t)$ are the dual basis functions of the Bernstein polynomials $B_{k-1}^n(t), \ldots, B_{n-k+1}^n(t)$ in $\mathcal{P}^{(n,k-1)}$. The coefficients $\alpha_j^{(k)}$ and $\beta_j^{(k)}$ satisfy the two equations

$$\begin{aligned} 0 &= c_{j,k-1}^{(k-1)} - \alpha_j^{(k)} c_{k-1,k-1}^{(k-1)} - \beta_j^{(k)} c_{n-k+1,k-1}^{(k-1)} \quad \text{and} \\ 0 &= c_{j,n-k+1}^{(k-1)} - \alpha_j^{(k)} c_{k-1,n-k+1}^{(k-1)} - \beta_j^{(k)} c_{n-k+1,n-k+1}^{(k-1)}. \end{aligned} \tag{23}$$

Therefore the functions $D_k^{(n,k)}(t), \ldots, D_{n-k}^{(n,k)}(t)$ obtained from (22) (or equivalently (20)) are contained in the polynomial space $\mathcal{P}^{(n,k)}$.

For the inner products $(B_i^n(t), D_j^{(n,k)}(t))$ $(i, j = k, \ldots, n-k)$ we get

$$(B_i^n, D_j^{(n,k)}) = \underbrace{(B_i^n, D_j^{(n,k-1)})}_{=\delta_{i,j}} - \alpha_j^{(k)} \underbrace{(B_i^n, D_{k-1}^{(n,k-1)})}_{=0} - \beta_j^{(k)} \underbrace{(B_i^n, D_{n-k+1}^{(n,k-1)})}_{=0} = \delta_{i,j}. \tag{24}$$

The basis of the induction is guaranteed by (19) and by Theorem 3. □

As an example, the Bernstein basis $\{B_3^9(t), \ldots, B_6^9(t)\}$ of the polynomial space $\mathcal{P}^{(9,3)}$ and its dual basis $\{D_3^{(9,3)}(t), \ldots, D_6^{(9,3)}(t)\}$ have been plotted in Figure 2.

4. Dual functionals

Using the dual basis functions $D_0^n(t), \ldots, D_n^n(t)$ one may introduce the linear functionals

$$\Delta_i^n : f(t) \mapsto \Big(D_i^n(t), f(t) \Big) \qquad (i = 0, \ldots, n). \tag{25}$$

These functionals apply to any quadratically integrable function $f(t) : [0,1] \to \mathbb{R}$. They are the dual functionals of the Bernstein polynomials as $\Delta_i^n(B_j^n(t)) = \delta_{i,j}$

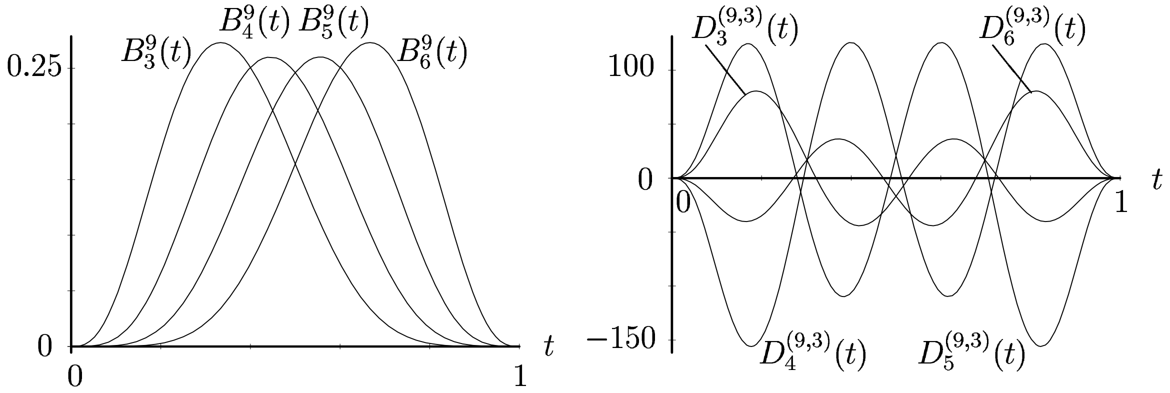

Figure 2: The Bernstein polynomials $B_3^9(t), \ldots, B_6^9(t)$ and the associated dual basis functions.

holds. When applied to a polynomial of maximal degree n, these functionals yield the coefficients of its Bernstein–Bézier representation.

Barry and Goldman [1, 2] have derived similar dual functionals from the de Boor/Fix–functionals (cf. [4], see also [5, 12] for the multivariate case). Both sets of dual functionals agree on the space $\mathcal{P}^n$ of polynomials of maximal degree n (that is, on the span of the functions where they are dual to). Their extensions to arbitrary functions, however, are different.

The dual functionals which are obtained from the de Boor/Fix–functionals can be applied to any function with existing n–th derivative. The functionals Δ_i^n, by contrast, can be applied to any quadratically integrable function. In addition, the functionals Δ_i^n, when applied to a function $f(t)$, produce the coefficients of the least–squares approximation of $f(t)$ in $\mathcal{P}^n$. That is, the polynomial

$$p(t) = \sum_{i=k}^{n-k} \Delta_i^n(f)\, B_i^n(t) \tag{26}$$

is the unique polynomial of degree n which minimizes

$$(f-p, f-p) = \int_0^1 (f(t)-p(t))^2 \,\mathrm{d}t. \tag{27}$$

The dual functionals which are obtained from the de Boor–Fix functionals, by contrast, lead to different coefficients in general.

With the help of the dual basis functions $D_i^{(n,k)}(t)$ one may easily find a similar formula for least–squares approximation with boundary conditions.

As the standard approach, the least–squares approximation of a function is computed by solving the linear system of the normal equations. This approach will be preferred in practice, particularly as it applies to more general classes of approximating functions like splines. As a well–known fact, it is advantageous to use splines rather than polynomials of higher degree, due to stability and efficiency reasons.

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