Abstract to

Visualization of moving objects using dual quaternion curves

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The interpolation of some positions (= point + orientation) of a moving object is examined with help of dual quaternion curves. In order to apply the powerful methods of Computer Aided Geometric Design, an interpolating motion whose trajectories are rational Bézier curves is constructed. The interpolation problem is discussed from a mechanical and a geometrical viewpoint. A representation formula for rational motions of fixed order is presented. Finally, the construction of rational spline motions is outlined. Dual quaternions prove to be very useful in Computer Graphics.

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Introduction

The present paper discusses the following interpolation problem: Let some positions (= points + orientations) of a moving object in 3–space be given. A continuous motion interpolating these positions is to be found.
The solution of this problem is required in Computer Graphics (in order to animate objects) as well as in Robotics (e.g., for the path planning of robot manipulators). Most of recently used algorithms solving this interpolation problem describe the orientations of the moving object by rotational parameters like Euler angles and interpolate these parameters, e.g. using spline functions. Then, the trajectories of the moving object are non–rational curves in general. They have to be computed using trigonometric functions.
Because of their fast and stable algorithms, the methods of Computer Aided Geometric Design (see [7]) seem to become more and more popular in Computer Graphics. In order to interpolate rotations using normalized quaternion curves, a spherical generalization of the de Casteljau–algorithm has been developed in [15], [11] and [12]. The orientation of the moving object is described by a normalized quaternion corresponding to a point on the unit sphere of 4–space, these points are interpolated by a curve generated by the spherical de Casteljau–algorithm.
The method has proved to be powerful, but the interpolating motion possesses some disadvantages: The trajectories of the moving object are non–rational curves. (In fact, their

explicit parametric representation seems to be unknown!) The interpolation of more than two positions by one motion and the construction of higher than first order continuous spline motions turn out to be difficult.

Another approach to the solution of the interpolation problem has been suggested by Ge and Ravani [6]. The positions of the moving objects are represented using dual quaternion curves without any normalization conditions. A multiplication of these curves with arbitrary *dual* factors does not change the described motion. A de Casteljau–like algorithm is formulated, but the influence of the weights of the control points (which are *dual* numbers!) is very complex. For example, the linear interpolation of two positions using Ge and Ravani's method is not unique. (It can not be the unique screw motion connecting both positions because the trajectories of screw motions (helices) are transcendental curves in general.)

In this paper, the positions of the moving object will be represented by dual quaternion curves satisfying a quadratic normalization condition (Plücker's condition). These curves can be multiplied with arbitrary real factors without influencing the described motion. They are described by Bézier curves, therefore the trajectories of the moving object are rational Bézier curves, too.

The use of *rational motions* (i.e., motions with rational trajectories) has some important advantages:

- The methods of Computer Aided Geometric Design can be applied *directly to the trajectories* of the moving object.
- The curves and surfaces generated by a rational motion are Bézier and B–spline curves and surfaces. Thus, the use of rational motions supports the data exchange with CAD systems.
- Rational motions can be applied to the numerical control of milling machines. This is advantageous for the generation of free–form surfaces described by rational Bézier and B–spline surfaces.
- Collision tests (which are important for the path planning of robot manipulators) prove to be equivalent to the computation of the roots of certain polynomials. (Note that collision tests can be performed using dual quaternions [5].)

At first, the paper briefly summarizes some fundamentals of the use of dual quaternions in spatial kinematics. The second section formulates the interpolation problem and outlines the choice of the parametrization of the given positions. Then, three methods for the interpolation of the orientations of the given positions are presented and compared in section 3. The next section derives two methods for the interpolation of the whole positions and discusses the dependence of the results on the choice of the coordinate systems. Section 5 outlines the interpolation with rational motions of minimal order, i.e., with motions whose trajectories are of minimal polynomial degree. The order of a rational motion has been considered in 1895 at first [4]. This paper presents a representation formula for rational motions of fixed order. Rational spline motions are briefly discussed in section 6. The final section summarizes the visualization algorithm.

1. Dual quaternions in spatial kinematics

Dual quaternions are a powerful tool in spatial kinematics. This section briefly summarizes the connection between spatial displacements and the non–commutative ring of dual quaternions. A detailed introduction to this topic can be found e.g. in [3] or [2].

Consider two Euclidean 3-spaces E^3 and $\widehat{\mathrm{E}}^3$ with Cartesian coordinate systems $\{\mathrm{O}, \vec{\mathbf{e}}_x, \vec{\mathbf{e}}_y, \vec{\mathbf{e}}_z\}$ and $\{\widehat{\mathrm{O}}, \widehat{\vec{\mathbf{e}}}_x, \widehat{\vec{\mathbf{e}}}_y, \widehat{\vec{\mathbf{e}}}_z\}$, respectively. In order to simplify notations, their points will be described by *homogeneous coordinate vectors* $\mathbf{x} = (\, x_0 \; x_1 \; x_2 \; x_3 \,)^\top$ from $\mathrm{I\!R}^4$. These coordinates are defined by the relation

$$x_0 : x_1 : x_2 : x_3 = 1 : x : y : z. \tag{1}$$

The 0-th coordinate is the homogenizing one. (It is often called the *weight* of the point.)

The 3-space $\widehat{\mathrm{E}}^3$ results from E^3 by an Euclidean *spatial displacement.* This displacement will be described with help of dual quaternions:

A Quaternion $Q^0 = q^0 + \vec{\mathbf{q}}^{\,0}$ consists of the scalar part $q^0 = \mathrm{Scal}\; Q^0 \in \mathrm{I\!R}$ and of the vector part $\vec{\mathbf{q}}^{\,0} = \mathrm{Vec}\; Q^0 \in \mathrm{I\!R}^3$. The set of quaternions with the componentwise addition and with the multiplication

$$(a^0 + \vec{\mathbf{a}}^{\,0}) * (b^0 + \vec{\mathbf{b}}^{\,0}) = (a^0 b^0 - \vec{\mathbf{a}}^{\,0} \circ \vec{\mathbf{b}}^{\,0}) + (a^0 \vec{\mathbf{b}}^{\,0} + b^0 \vec{\mathbf{a}}^{\,0} + \vec{\mathbf{a}}^{\,0} \times \vec{\mathbf{b}}^{\,0}) \tag{2}$$

(where $\circ$ and $\times$ denote the inner and the cross product of vectors from $\mathrm{I\!R}^3$, respectively) forms the skew field $\mathrm{I\!H}$. The adjunction of the *dual unit* ε with $\varepsilon^2 = 0$ yields the non–commutative ring $\mathrm{I\!H}[\varepsilon]$ of *dual quaternions.* A dual quaternion

$$Q = Q^0 + \varepsilon Q^\varepsilon = (q^0 + \vec{\mathbf{q}}^{\,0}) + \varepsilon(q^\varepsilon + \vec{\mathbf{q}}^{\,\varepsilon}) \tag{3}$$

consists of the real part $Q^0 = \mathrm{Re}\; Q \in \mathrm{I\!H}$ and of the dual part $Q^\varepsilon = \mathrm{Du}\; Q \in \mathrm{I\!H}$. The conjugate dual quaternion of (3) is

$$\widetilde{Q} = \widetilde{Q}^0 + \varepsilon \widetilde{Q}^\varepsilon = (q^0 - \vec{\mathbf{q}}^{\,0}) + \varepsilon(q^\varepsilon - \vec{\mathbf{q}}^{\,\varepsilon}). \tag{4}$$

The dual quaternion

$$2v_0 + \varepsilon \vec{\mathbf{v}} \qquad (v_0 \in \mathrm{I\!R},\ \vec{\mathbf{v}} \in \mathrm{I\!R}^3) \tag{5}$$

corresponds to the *translation* with the displacement vector $\frac{1}{v_0}\vec{\mathbf{v}}$ $(v_0 \neq 0)$. The quaternion $D = d_0 + \vec{\mathbf{d}} \in \mathrm{I\!H}$ $(D \neq 0)$ describes a *rotation* around the origin. The normalized direction vector $\vec{\mathbf{r}}$ of the axis $(\mathrm{O}, \vec{\mathbf{r}})$ as well as the angle φ of the rotation can be found from

$$(d_0 + \vec{\mathbf{d}}) = \sqrt{d_0^2 + \vec{\mathbf{d}} \circ \vec{\mathbf{d}}}\; \underbrace{(\cos\tfrac{\varphi}{2} + \sin\tfrac{\varphi}{2}\vec{\mathbf{r}})}_{=:\, E} \qquad (||\vec{\mathbf{r}}|| = 1). \tag{6}$$

The four components of the *normalized* quaternion E (i.e., $E * \widetilde{E} = 1$) are called *Euler's parameters* of the rotation.
A *spatial displacement* is the composition of the translation (5) with the rotation (6). It corresponds to the dual quaternion

$$Q = Q^0 + Q^\varepsilon = \underbrace{(2v_0 + \varepsilon\vec{\mathbf{v}})}_{\mathrm{Trans}(Q)} * \underbrace{(d_0 + \vec{\mathbf{d}})}_{\mathrm{Rot}(Q)} = \underbrace{2v_0 d_0}_{=q^0} + \underbrace{2v_0\vec{\mathbf{d}}}_{=\vec{\mathbf{q}}^0} + \varepsilon(\underbrace{-\vec{\mathbf{v}}\circ\vec{\mathbf{d}}}_{=q^\varepsilon} + \underbrace{d_0\vec{\mathbf{v}} + \vec{\mathbf{v}}\times\vec{\mathbf{d}}}_{=\vec{\mathbf{q}}^\varepsilon}). \tag{7}$$

This dual quaternion satisfies *Plücker's relation*

$$\mathrm{Du}\ (Q * \widetilde{Q}) = q^0 q^\varepsilon + \vec{\mathbf{q}}^0 \circ \vec{\mathbf{q}}^\varepsilon = 0. \tag{8}$$

On the other hand, any dual quaternion Q satisfying (8), whose real part does not vanish, describes a spatial displacement. Its translational and (normalized) rotational parts are

$$\mathrm{Trans}(Q) = Q * \widetilde{Q}^0 \quad \text{and} \quad \mathrm{Rot}(Q) = \frac{1}{Q^0 * \widetilde{Q}^0}\, Q^0, \tag{9}$$

respectively. *Proportional dual quaternions* (i.e., quaternions which differ only by a real factor) *describe the same spatial displacement. The multiplication of dual quaternions satisfying (8) corresponds to the composition of spatial displacements.*
The Euclidean space $\widehat{\mathrm{E}}^3$ was assumed to result from E^3 by the spatial displacement (7). Consider a point $\mathbf{x} \in \widehat{\mathrm{E}}^3$. Its homogeneous coordinates with respect to E^3 can be computed with help of the transformation $\mathrm{M} : \mathbf{x} \in \widehat{\mathrm{E}}^3 \mapsto \mathrm{M}\mathbf{x} \in \mathrm{E}^3$, where M denotes the real 4×4–matrix

$$\mathrm{M} = \left(\begin{array}{c|ccc} (d_0^2 + \vec{\mathbf{d}}\circ\vec{\mathbf{d}})v_0 & 0 & 0 & 0 \\ \hline (d_0^2 + \vec{\mathbf{d}}\circ\vec{\mathbf{d}})\vec{\mathbf{v}} & & v_0\,\mathrm{U} & \end{array}\right). \tag{10}$$

The orthogonal 3×3–matrix U describes the rotational part of the spatial displacement (7):

$$\mathrm{U} = \begin{pmatrix} d_0^2 + d_1^2 - d_2^2 - d_3^2 & 2(d_1 d_2 - d_0 d_3) & 2(d_0 d_2 + d_1 d_3) \\ 2(d_0 d_3 + d_1 d_2) & d_0^2 + d_2^2 - d_1^2 - d_3^2 & 2(d_2 d_3 - d_0 d_1) \\ 2(d_1 d_3 - d_0 d_2) & 2(d_0 d_1 + d_2 d_3) & d_0^2 + d_3^2 - d_1^2 - d_2^2 \end{pmatrix} \tag{11}$$

$$(\text{with} \quad \vec{\mathbf{d}} = (\, d_1\ d_2\ d_3\,)^\top).$$

The homogeneous coordinates $\mathbf{y} = (\, y_0\ \vec{\mathbf{y}}^\top\,)^\top$ of the point $\mathbf{x} = (\, x_0\ \vec{\mathbf{x}}^\top\,)^\top \in \widehat{\mathrm{E}}^3$ with respect to E^3 can be computed with help of dual quaternions, too:

$$y_0 = (Q^0 * \widetilde{Q}^0)x_0 \quad \text{and} \quad \vec{\mathbf{y}} = \mathrm{Du\ Vec}\ (\,Q * [\,2x_0 + \varepsilon\,\vec{\mathbf{x}}\,] * \widetilde{Q}^0\,). \tag{12}$$

A continuous series of spatial displacements is called a *motion.* The Euclidean spaces E^3 and $\widehat{\mathrm{E}}^3$ are called the *fixed* and the *moving space* of this motion, respectively. The motion is described either by a dual quaternion curve $Q = Q(t)$ satisfying Plücker's relation or by a matrix-valued function $\mathrm{M} = \mathrm{M}(t)$, cf. (10). The parameter $t \in \mathbb{R}$ may be identified with the

time. The images of a point $\mathbf{x}$ of the moving space $\widehat{\mathrm{E}}^3$ under the transformations $\mathrm{M} = \mathrm{M}(t)$ form the curve $\mathbf{y}(t) = \mathrm{M}(t)\,\mathbf{x}$ in the fixed space. This curve is called the *trajectory* or the *path* of the point.
Resulting from its description as a dual quaternion curve, a motion can be considered as a curve on the quadric hypersurface (8) of the real projective 7–space. This paper focuses on rational motions described by dual quaternions. In order to apply the powerful methods of Computer Aided Geometric Design (see e.g. [7]), the Bernstein–Bézier–technique will be used:

Definition. Consider the polynomial

$$Q(t) = \sum_{i=0}^{n} b_i^n(t) B_i \quad t \in \mathbb{R} \tag{13}$$

(where the $b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are the Bernstein polynomials) with coefficients $B_i \in \mathbb{H}[\varepsilon]$. If this polynomial satisfies Plückers relation (8), then the corresponding motion is called a *Q–motion* of degree n.

Note that the trajectories of the points of the moving space by a Q–motion of degree n are rational Bézier curves of order $2n$ in general, see (12). The connection between the order of the trajectories and the degree of the Q–motion has been thoroughly discussed in [9]. This paper presents a construction for interpolating Q–motions:

2. The interpolation problem

Let $m+1$ spatial displacements

$$P_i = \underbrace{(2 + \varepsilon\,\vec{\mathbf{s}}_i)}_{\mathrm{Trans}(P_i)} * \underbrace{(r_{i,0} + \vec{\mathbf{r}}_i)}_{\mathrm{Rot}(P_i)} \quad (\vec{\mathbf{s}}_i,\, \vec{\mathbf{r}}_i \in \mathbb{R}^3;\ \mathrm{r}_{\mathrm{i},0} \in \mathbb{R}) \tag{14}$$

(cf. (7)) with parameters $t_i \in \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_m = 1$ be given ($i = 0, \ldots, m$). The displacement P_i is assumed to describe the position of the moving space $\widehat{\mathrm{E}}^3$ with respect to the fixed space E^3 at the point of time t_i.
If the parameters t_i are unknown, then they can be estimated with help of the distances and the angles between the given positions. The abbreviations $\mathrm{dist}(P_i, P_{i+1})$ and $\sphericalangle(P_i, P_{i+1})$ denote the distance between the origins of the positions P_i, P_{i+1} and the angle of the rotation connecting the rotational part of P_i with that of P_{i+1}, respectively. Note that $\mathrm{dist}(P_i, P_{i+1})$ depends on the choice of the origin of the moving space $\widehat{\mathrm{E}}^3$! In order to avoid this dependence, the unique screw motion connecting the positions P_i and P_{i+1} is considered, cf. [3, p.35]. The length of the displacement vector of this screw motion is denoted by $\mathrm{dist}^*(P_i, P_{i+1})$. This length can be computed by projecting the vector from the origin of P_i to that of P_{i+1} onto the axis of the rotation, which connects the rotational parts of the two positions.

Some possibilities for the choice of the parametrization are ($\triangle t_i = t_{i+1} - t_i$):

$$\begin{array}{lcll} \triangle t_i & \sim & \mathrm{dist}(P_i, P_{i+1}) & \text{(chordal)}, \\ \triangle t_i & \sim & \mathrm{dist}^*(P_i, P_{i+1}) & \text{($*$-chordal)}, \\ \triangle t_i & = & \text{constant} & \text{(equidistant)}, \\ \triangle t_i & \sim & \sqrt{\mathrm{dist}(P_i, P_{i+1})} & \text{(centripetal)}, \\ \triangle t_i & \sim & \sphericalangle(P_i, P_{i+1}), & \text{(angular)} \\ \triangle t_i & \sim & \sphericalangle(P_i, P_{i+1}) + \mathrm{dist}(P_i, P_{i+1}), & \\ & \vdots & & \end{array} \tag{15}$$

The $m+1$ positions (14) are to be interpolated by a Q–motion of degree n, i.e., a Q–motion (13) satisfying

$$Q(t_i) = \lambda_i P_i \quad (i = 0, \ldots, m) \tag{16}$$

is to be found. The real factors $\lambda_i \neq 0$ are arbitrary.
The interpolation scheme should fulfill the following requirements:

(i) The interpolating Q–motion is found by solving a system of linear equations.

(ii) The result of the interpolation scheme does not depend on the choice of the orientations of the fixed and of the moving space. I.e., applying the interpolation scheme to the positions $P_i * D$ resp. $D * P_i$ (which result from the original positions by a fixed rotation $D \in \mathbb{H}$ of the moving resp. of the fixed space) yields the Q–motion $Q(t) * D$ resp. $D * Q(t)$, where $Q(t)$ denotes the result of the interpolation of the original positions.

(iii) Furthermore, the interpolation problem will be considered from two different viewpoints: From a *mechanical* viewpoint, the interpolating Q–motion describes the motion of a rigid body, and the origin of the moving space has a special meaning, e.g. it is the centre of gravity. In this case, the result of the interpolation method should depend on the choice of the origin of the moving space.
From a *geometrical* viewpoint, the interpolating Q–motion only describes the motion of the moving space with respect to the fixed space, and the result of the interpolation scheme should be independent on the choice of the origins of the coordinate systems.

Some notations used in the remaining sections are summarized in the following table:

$P_i = \underbrace{(2+\varepsilon\,\vec{\mathbf{s}}_i)}_{\mathrm{Trans}(P_i)} * \underbrace{(r_{i,0}+\vec{\mathbf{r}}_i)}_{\mathrm{Rot}(P_i)} \in \mathbb{H}[\varepsilon]$	given positions of the moving object
$t_i \in [0,1]$	their parameters $(i = 0, \ldots, m)$
$Q = Q(t)$	interpolating Q–motion
$Q_{\mathrm{rot}} = Q_{\mathrm{rot}}(t)$	rotational part of $Q(t)$ (see Section 3)
$C_j \in \mathbb{H}$	coefficients of $Q_{\mathrm{rot}}(t)$, see (17) $(j = 0, \ldots, k)$
$Q^{[(1/2)]}_{\mathrm{trans}} = Q^{[(1/2)]}_{\mathrm{trans}}(t)$	translational part(s) of $Q(t)$ (see Section 4)
$\vec{\mathbf{p}}^{[(1/2)]}_j \in \mathbb{R}^3$	coefficients of $Q^{[(1/2)]}_{\mathrm{trans}}(t)$, see (30) and (32) $(j = 0, \ldots, l_{[1/2]})$

3. The interpolation of the rotational part

At first, the given orientations of the moving space will be interpolated by a rotational Q–motion. Later, this motion is composed with appropriate translational motions.
The rotational part of the interpolating Q–motion has the form

$$Q_{\mathrm{rot}}(t) = \sum_{j=0}^{k} b^k_j(t) C_j, \tag{17}$$

the coefficients $C_j \in \mathbb{H}$ are unknown. The motion (17) has to satisfy the interpolation conditions

$$Q_{\mathrm{rot}}(t_i) = \lambda_i \underbrace{(r_{i,0} + \vec{\mathbf{r}}_i)}_{=\mathrm{Rot}(P_i)} \quad (i = 0, \ldots, m) \tag{18}$$

with arbitrary real factors $\lambda_i \neq 0$. Three different methods for the interpolation of the rotational part are presented:

3.1 The affine method

This method is based on the normalization

$$\mathrm{Scal}\; Q_{\mathrm{rot}}(t) \equiv 1 \tag{19}$$

of the interpolating rotational Q–motion (17). (This normalization yields the Rodrigues' parameter of the rotation, see [1, p.19].) Then the real factors λ_i from (18) have to satisfy

$$\lambda_i = \frac{1}{r_{i,0}} \quad (r_{i,0} \neq 0) \tag{20}$$

and the interpolation conditions (18) yield the equations

$$\begin{aligned} &\sum_{j=0}^{k} b_j^k(t_i)\,\mathrm{Vec}\,(C_j) = \frac{1}{r_{i,0}}\vec{\mathbf{r}}_i && (i = 0,\dots,m) \\ \text{and} \quad &\mathrm{Scal}\,(C_j) = 1 && (j = 0,\dots,k) \end{aligned} \tag{21}$$

They form a linear system for the unknown coefficients $C_j \in \mathbb{H}$. If $k = m$ holds, then a unique solution exists.

The normalization (19) introduces in the space $\mathbb{H}$ of quaternions the structure of an affine 3–space. Positions with $r_{i,0} = 0$ (i.e., positions which result from the fixed space by a rotation with angle $\varphi = \pi$) are *at infinity*, they cannot be interpolated. Of course, the set of positions at infinity depends on the choice of the orientation of the fixed space E^3. Thus, the affine interpolation method does not fulfill the requirement (ii): its result depends on the choice of the orientation of the fixed space.

Proposition 1. *The rotational parts of the $m + 1$ given positions (14) can be interpolated by a unique Q–motion (17) of degree $k = m$ satisfying the normalization condition (19). Its coefficients result from the system (21) of linear equations. Positions with $r_{i,0} = 0$ cannot be interpolated. The result of the interpolation method depends on the choice of the orientation of the fixed space.*

The next method uses another normalization and avoids the disadvantage of the affine method.

3.2 The elliptic method

The elliptic method uses the normalization

$$Q_{\mathrm{rot}}(t_i) * \widetilde{Q}_{\mathrm{rot}}(t_i) = d_0(t_i)^2 + \vec{\mathbf{d}}(t_i) \circ \vec{\mathbf{d}}(t_i) = 1 \quad (i = 0,\dots,m), \tag{22}$$

the points $Q_{\mathrm{rot}}(t_i) = d_0(t_i) + \vec{\mathbf{d}}(t_i)$ lie on the *unit sphere* of $\mathbb{H}$. Let $R_i^{(0)}$ be the normalized quaternions obtained from $\mathrm{Rot}(P_i)$:

$$R_i^{(0)} = \underbrace{\pm\frac{1}{\sqrt{r_{i,0}^2 + \vec{\mathbf{r}}_i \circ \vec{\mathbf{r}}_i}}}_{=\lambda_i}\,(r_{i,0} + \vec{\mathbf{r}}_i). \quad (i = 0,\dots,m) \tag{23}$$

The signs of the factors λ_i have to be chosen – corresponding to the choice between point $R_i^{(0)}$ and antipodal point $-R_i^{(0)}$ on the unit sphere of $\mathbb{H}$. The following considerations assume, that the signs of λ_i satisfy

$$(R_i^{(0)} - R_{i+1}^{(0)}) * (\widetilde{R}_i^{(0)} - \widetilde{R}_{i+1}^{(0)}) \le (R_i^{(0)} + R_{i+1}^{(0)}) * (\widetilde{R}_i^{(0)} + \widetilde{R}_{i+1}^{(0)}) \quad (i = 0,\dots,m-1), \tag{24}$$

i.e., that the Euclidean distances in the 4–space $\mathbb{H}$ between adjacent normalized quaternions $R_i^{(0)}$ and $R_{i+1}^{(0)}$ are minimized. The sign of the first position $R_0^{(0)}$ is arbitrary, the remaining

signs are determined by (24).
The interpolation conditions (18) yield the equations

$$\sum_{j=0}^{k} b_j^k(t_i)\, C_j = R_i^{(0)} \quad (i = 0, \ldots, m) \tag{25}$$

They form a linear system for the unknown coefficients $C_j \in \mathbb{H}$. If $k = m$ holds, then a unique solution exists:

Proposition 2. *The rotational parts of the $m + 1$ given positions (14) can be interpolated by a unique Q–motion (17) of degree $k = m$ satisfying the interpolation conditions (25). The result of the interpolation method does not depend on the choice of the orientations of the fixed space and of the moving space.*

Proof. The independence of the choice of the orientations remains to be shown. Consider a fixed rotation $D \in \mathbb{H}$ of the fixed space. This rotation corresponds to the transformation $R_i \mapsto D * R_i$ of the rotational parts of the given positions ($R_i = \mathrm{Rot}(P_i)$). Let without loss of generality $D * \widetilde{D} = 1$ be assumed. The transformed Q–motion $D * Q_{\mathrm{rot}}(t)$ satisfies the normalization condition (22):

$$D * \underbrace{Q_{\mathrm{rot}}(t_i) * \widetilde{Q}_{\mathrm{rot}}(t_i)}_{=1} * \widetilde{D} = 1 \quad (i = 0, \ldots, m), \tag{26}$$

thus applying the elliptic interpolation scheme to the transformed positions $D * R_i$ yields the Q–motion $D * Q_{\mathrm{rot}}(t)$. Similar considerations prove the assertion in the case of a fixed rotation of the moving space. ■

The choice of another orientation of the fixed resp. of the moving space corresponds to a rotation of the 4–space $\mathbb{H}$ around the origin, this rotation maps the unit sphere in $\mathbb{H}$ to itself. The result of the elliptic interpolation scheme is invariant with respect to these rotations. (Note that also the normalization (24) is preserved by these rotations!)
Consider the set $\mathbb{H}$ of quaternions as a real projective 3–space. The rotations of the 4–space $\mathbb{H}$ around its origin form the group of all *elliptic transformations* of the projective 3–space $\mathbb{H}$. *Thus, the normalization (22) induces in $\mathbb{H}$ the structure of an elliptic 3–space.* This explains the designation of the interpolation method.
Figure 1a) shows a rotational Q–motion of degree 4 interpolating 5 given positions with equidistant parameters t_i. This motion has been computed with help of the elliptic method. In order to obtain a nonambiguous picture, the rotational motion $Q_{\mathrm{rot}}(t)$ has been composed with an appropriate translational motion. The trajectory of the origin of the moving space has been drawn. In Figure 1b) some positions of the unit cube of the moving space are shown. The three pairs of opposite faces of the cube are marked by squares, triangles and crosses, respectively.

Figure 1: Interpolation of the rotational part using the elliptic method.

3.3 The projective method

This method dispenses with any normalization. If $Q_{\text{rot}}(t_i) \neq 0$ holds, then the interpolation (18) is equivalent to

$$r_{i,0}\ \text{Vec}\ (Q_{\text{rot}}(t_i)) - \text{Scal}\ (Q_{\text{rot}}(t_i))\ \vec{\mathbf{r}}_i \quad = \quad 0 \quad \text{resp.} \tag{27}$$

$$\sum_{j=0}^{k} b_j^k(t_i)\ [r_{i,0}\ \text{Vec}\ (C_j) - \text{Scal}\ (C_j)\ \vec{\mathbf{r}}_i] \quad = \quad 0 \quad (i = 0, \ldots, m). \tag{28}$$

The $m+1$ equations (28) form a homogeneous linear system for the unknown coefficients C_j. Let $k = \frac{3}{4}m$ be assumed. Then at least one non–trivial solution of the system exists. Additionally, this solution is unique up to real factors, see [8].
If $Q(t_i) = 0$ holds, then the given position P_i is *inaccessible* in the sense of numerical analysis. These positions have to be excluded.
Consider again the set $\mathbb{H}$ of quaternions as a real projective 3–space. The result of the interpolation method is even invariant with respect to *projective* transformations, because the interpolation conditions have been formulated in a completely invariant way. Especially, the result of the projective interpolation method does not depend on the choice of the orientations of the fixed and of the moving space: the elliptic transformations of an projective 3–space form a subgroup of the projective transformations. Thus we have:

Proposition 3. *The rotational parts of the $m+1$ given positions (14) can be interpolated by a Q–motion (17) of degree $k = \frac{3}{4}m$ satisfying the interpolation conditions (27). This motion is unique up to real factors. The result of the interpolation method does not depend on the choice of the orientations of the fixed space and of the moving space.*

As an important disadvantage, the projective method often produces "somersaults" of the interpolating Q–motion. These somersaults occur, if the curve $Q_{\text{rot}}(t)$ comes very near to the origin $0 \in \mathbb{H}$. This disadvantage results from the fact, that the result of the method is invariant with respect to the group of projective transformations of $\mathbb{H}$. This group is too large, it is not the appropriate one for the rotational interpolation problem.
Figure 2a) shows a rotational Q–motion of degree 3 interpolating 5 given positions with equidistant parameters t_i. This motion has been computed with help of the projective method, and it has been composed with an appropriate translational motion. The trajectories of the origin and of the point $(0\ 0\ 1)^\top$ of the moving space have been drawn. The projective interpolation method has produced one "somersault" of the interpolating motion. Some positions of the unit cube of the moving space are shown in Figure 2b).

Figure 2: Interpolation of the rotational part using the projective method.

Comparing the results of the three methods, one obtains that the elliptic interpolation method yields the best Q–motions interpolating the rotational parts of the given positions. Another obvious idea is to consider completely normalized motions $Q_{\mathrm{rot}}(t)$, i.e., quaternion curves on the unit sphere of $\mathbb{H}$. But the use of rational curves on the unit sphere of $\mathbb{H}$ would nearly produce a doubling of the required polynomial degrees of the trajectories. Furthermore, the normalization does not possess a direct geometrical meaning: the trajectories of a non–normalized Q–motion are rational too. Therefore this paper will not use this normalization.

4. Interpolation with Q–motions

In this section we assume that the rotational parts of the given positions (14) are interpolated by a rotational Q–motion $Q_{\mathrm{rot}}(t)$ of degree k, cf. (17). Now, this rotational motion is composed with appropriate translational motions in order to interpolate the whole positions. The coefficients $B_i \in \mathbb{H}[\varepsilon]$ of the interpolating motion $Q(t)$ (cf. (13)) can be computed with help of product formulae like

$$b_i^k(t) b_j^l(t) = \frac{\binom{k}{i}\binom{l}{j}}{\binom{k+l}{i+j}} b_{i+j}^{k+l}(t). \tag{29}$$

4.1 The TR–method

This method composes the rotational motion $Q_{\mathrm{rot}}(t)$ with a single translational Q–motion $Q_{\mathrm{trans}}(t)$ of degree l:

$$\begin{aligned} Q(t) &= Q_{\mathrm{trans}}(t) * Q_{\mathrm{rot}}(t) \\ &= [\,2 + \varepsilon \sum_{j=0}^{l} b_j^l(t)\,\vec{\mathbf{p}}_j\,] * Q_{\mathrm{rot}}(t). \end{aligned} \tag{30}$$

In order to avoid poles, the scalar part of $Q_{\mathrm{trans}}(t)$ is normalized to 2. The coefficients $\vec{\mathbf{p}}_j \in \mathbb{R}^3$ are unknown. The motion (30) has to interpolate the given spatial displacements (14). The interpolation conditions (16) yield the equations

$$\sum_{j=0}^{l} b_j^l(t_i)\,\vec{\mathbf{p}}_j = \vec{\mathbf{s}}_i \quad (i = 0, \ldots, m). \tag{31}$$

They form a linear system for the unknown coefficients $\vec{\mathbf{p}}_j \in \mathbb{R}^3$. If $l = m$ holds, then a unique solution exists:

Proposition 4. *The $m+1$ given positions (14) can be interpolated by a unique Q–motion (30) of degree $k+m$ (where $l = m$). In general, the result of the TR–method depends on the choice of the origin of the moving space.*

Proof. The trajectory of the origin of the moving space $\widehat{\mathrm{E}}^3$ is a rational curve of degree $l = m$, whereas the trajectory of an arbitrary point of the moving space is of degree $2k + l$ in general (see (10)). Thus the result of the TR–method depends on the choice of the origin of the moving space. ■

Figure 3a) shows a Q–motion of degree 8 interpolating 5 given positions obtained from the TR–method ($l = 4$). The parameters t_i have been chosen according to $\triangle t_i \sim \sqrt{\mathrm{dist}(P_i, P_{i+1})}$, the rotational parts of the given positions have been interpolated using the elliptic method ($k = 4$). The trajectories of the origin and of the point $(\,0\;0\;1\,)^\top$ of the moving space have been drawn. Figure 3b) shows some positions of the unit cube of the moving space.

Figure 3: Interpolation with the TR–method.

The combination of the TR–method with the elliptic scheme from the previous section is summarized in the following

Algorithm 1.

- Given: $m + 1$ positions P_i with parameters t_i, see (14) $(i = 0, \ldots, m)$.

1.) Normalize the rotational parts of the given positions corresponding to (23)! Choose the signs according to (24)!

2.) Compute the $m + 1$ coefficients $C_j \in \mathbb{H}$ of the rotational motion (17) by solving the system (25) of linear equations ($k = m$)!

3.) Compute the $m + 1$ coefficients $\vec{\mathbf{p}}_j \in \mathbb{R}^3$ of the translational motion $Q_{\mathrm{trans}}(t)$ (see (30)) by solving the system (31) of linear equations ($l = m$)!

4.) Multiply the translational and the rotational part, see (30)! The $2m + 1$ coefficients $B_j \in \mathbb{H}[\varepsilon]$ of the Q–motion (13) ($n = 2m$) result with help of the product formula (29).

4.2 The TRT–method

The TRT–method avoids the dependence on the choice of the origin. The rotational motion $Q_{\mathrm{rot}}(t)$ of degree k is composed with two translational Q–motions $Q^{(1)}_{\mathrm{trans}}(t)$ and $Q^{(2)}_{\mathrm{trans}}(t)$ of degree l_1 and l_2, respectively:

$$\begin{aligned} Q(t) &= Q^{(1)}_{\mathrm{trans}}(t) * Q_{\mathrm{rot}}(t) * Q^{(2)}_{\mathrm{trans}}(t) \\ &= [\,2 + \varepsilon \sum_{j_1=0}^{l_1} b^{l_1}_{j_1}(t)\,\vec{\mathbf{p}}^{(1)}_{j_1}\,] * Q_{\mathrm{rot}}(t) * [\,2 + \varepsilon \sum_{j_2=0}^{l_2} b^{l_2}_{j_2}(t)\,\vec{\mathbf{p}}^{(2)}_{j_2}\,]. \end{aligned} \qquad (32)$$

(Note that this motion is only of polynomial degree $\max(l_1, l_2) + k$ as $\varepsilon^2 = 0$ holds!) The coefficients $\vec{\mathbf{p}}_{j_1} \in \mathbb{R}^3$ and $\vec{\mathbf{p}}_{j_2} \in \mathbb{R}^3$ are unknown. The translational part of the motion is

$$Q^{(1)}_{\text{trans}}(t) * Q_{\text{rot}}(t) * Q^{(2)}_{\text{trans}}(t) * \widetilde{Q}_{\text{rot}}(t). \tag{33}$$

A short calculation leads to the interpolation conditions

$$(Q_{\text{rot}}(t_i) * \widetilde{Q}_{\text{rot}}(t_i)) \sum_{j_1=0}^{l_1} b^{l_1}_{j_1}(t_i)\, \vec{\mathbf{p}}^{(1)}_{j_1} \; + \; \sum_{j_2=0}^{l_2} b^{l_2}_{j_2}(t_i)\, \mathrm{U}(t_i) \vec{\mathbf{p}}^{(2)}_{j_2} = (Q_{\text{rot}}(t_i) * \widetilde{Q}_{\text{rot}}(t_i))\, \vec{\mathbf{s}}_i \tag{34}$$

($i = 0, \ldots, m$), where the orthogonal 3×3–matrix $\mathrm{U} = \mathrm{U}(t)$ is defined like in (11) with $Q_{\text{rot}}(t_i) = d_0 + \vec{\mathbf{d}}$. These equations form a linear system for the $l_1 + l_2 + 2$ unknown coefficients $\vec{\mathbf{p}}^{(1)}_{j_1}, \vec{\mathbf{p}}^{(2)}_{j_2} \in \mathbb{R}^3$. If $l_1 + l_2 = m - 1$ holds, then in general a unique solution exists. (But, for instance, if some of the given positions have the same orientation, then the matrix of the system (34) may not have maximal rank!) The interpolating Q–motion (32) does not depend on the choice of the origin of the moving space: a fixed translation of the moving space, i.e., a multiplication of (32) with a quaternion $1 + \varepsilon\, \vec{\mathbf{a}}$ ($\vec{\mathbf{a}} \in \mathbb{R}^3$), yields again a Q–motion of the form (32). Thus we have:

Result 5. *Let two integers $l_1, l_2 \leq 0$ satisfying $l_1 + l_2 = m - 1$ be given. In general the $m+1$ given positions (14) can be interpolated by a unique Q–motion (32) of degree $\max(l_1, l_2) + k$. The result of the TRT–method does not depend on the choice of the origin of the moving space.*

Note that the degree of the Q–motion (32) interpolating the given positions is less than that of the Q–motion (30).

If the TRT–method is used in combination with the elliptic or with the projective method of the previous section, then the result of the interpolation scheme fulfills the three requirements (i), (ii) and (iii). It depends only on the relative positions of the fixed with respect to the moving space.

Figure 4a) shows a Q–motion of degree 6 interpolating 5 given positions obtained from the TRT–method ($l_1 = 3, l_2 = 0$). The parameters t_i have been chosen according to $\triangle t_i \sim \mathrm{dist}(P_i, P_{i+1}) + \sphericalangle(P_i, P_{i+1})$, the rotational parts of the given positions have been interpolated using the elliptic method ($k = 4$). The trajectories of the origin and of the point $(\,0\;0\;1\,)^\top$ of the moving space have been drawn. Figure 4b) shows some positions of the unit cube of the moving space.

Figure 4: Interpolation with the TRT–method.

The TRT–method should be used, if the origin of the moving space does not have a special meaning, for example, if the dimensions of the moved object are very large in comparison with the distances between the given positions. Otherwise, the interpolating Q–motion should be calculated with help of the TR–method.

5. Interpolation with rational motions of minimal order

The use of dual quaternions yields a very compact description of spatial displacements: any position can be described by 8 real numbers. In contrast with this, the matrix representation (10) of a spatial displacement requires 13 essential numbers.
On the other hand, the polynomial degrees of the trajectories of the points from the moving space $\widehat{\mathrm{E}}^3$ by a Q–motion of degree n are equal to $2n$ in general. This very high degree is disadvantageous for certain applications, e.g. for the construction of *sweeping surfaces.*
The maximal polynomial degree of the trajectories (which are assumed to be irreducible, i.e., their homogeneous coordinates do not have any common linear factors) is called the *order* of a rational motion. The following theorem yields a representation formula for rational motions of fixed order:

Theorem 6. *Let $w_0^*(t)$, $w_i(t)$ and $d_j(t)$ be polynomials of maximal degree $n-2k, n$, and k, respectively ($i = 1,2,3$; $j = 0,1,2,3$), where the number k satisfies $0 \le k \le \frac{n}{2}$. Then the matrix–valued polynomial*

$$\mathrm{M}(t) = \left(\begin{array}{c|ccc} (d_0(t)^2 + d_1(t)^2 + d_2(t)^2 + d_3(t)^2) w_0^*(t) & 0 & 0 & 0 \\ \hline w_1(t) & & & \\ w_2(t) & & w_0^*(t)\,\mathrm{U}(t) & \\ w_3(t) & & & \end{array} \right). \tag{35}$$

of degree n (where the orthogonal 3×3–matrix $\mathrm{U}(t)$ is defined in (11)) describes a rational motion of order n.
Conversely, let a rational motion of order n by its matrix representation $\mathrm{R} = \mathrm{R}(t)$ be given. Then a number k satisfying $0 \le k \le \frac{n}{2}$ and 8 polynomials $w_0^*(t)$, $w_i(t)$ and $d_j(t)$ of the above maximal degrees exist, such that the matrix–valued polynomial $\mathrm{M}(t)$ (see (35)) differs from $\mathrm{R}(t)$ by a factor $\xi(t) \in \mathbb{R}$ at most, i.e., $\mathrm{R}(t) = \xi(t)\,\mathrm{M}(t)$ holds.

The first part of this theorem results from straightforward calculations. The second part of the theorem has been derived in [9]. The details of its proof are omitted here. A detailed geometrical discussion of rational motions of order $n \le 4$ can be found in [16] and [13].
The rational motion (35) of order n corresponds to the Q–motion

$$Q(t) = \underbrace{[\,2w_0^*(d_0^2 + d_1^2 + d_2^2 + d_3^2) + \varepsilon \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}]}_{=:\,Q_{\mathrm{trans}}(t)} * \underbrace{[\,d_0 + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}]}_{=:\,Q_{\mathrm{rot}}(t)} \tag{36}$$

of degree $k+n$. The TR–method for interpolation with Q–motions can be formulated for interpolation with rational motions of minimal order: The rotational parts of the given positions can be interpolated with any one of the three methods of section 3, for example with help of the elliptic method ($k = m$). The translational part is interpolated as follows. Let

$$w_0^*(t) \equiv 1 \quad \text{and} \quad \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} = \sum_{j=0}^{l} b_j^l(t)\, \vec{\mathbf{p}}_j, \tag{37}$$

i.e.,

$$Q_{\text{trans}}(t) = 2 \underbrace{(d_0(t)^2 + d_1(t)^2 + d_2(t)^2 + d_3(t)^2)}_{Q_{\text{rot}}(t) \,*\, \widetilde{Q}_{\text{rot}}(t)} + \varepsilon \sum_{j=0}^{l} b_j^l(t)\, \vec{\mathbf{p}}_j \tag{38}$$

(In order to avoid poles, $w_0^* \equiv 1$ is chosen.) The coefficients $\vec{\mathbf{p}}_j \in \mathbb{R}^3$ are unknown. The interpolation conditions yields the system

$$\sum_{j=0}^{l} b_j^l(t_i)\, \vec{\mathbf{p}}_j = (Q_{\text{rot}}(t_i) * \widetilde{Q}_{\text{rot}}(t_i))\vec{\mathbf{s}}_i \quad (i = 0, \ldots, m). \tag{39}$$

of linear equations. The coefficients $\vec{\mathbf{p}}_j \in \mathbb{R}^3$ are computed by solving the system (39). If $l = m$ holds, then a unique solution exists. The interpolating Q–motion (36) has the degree $3m$, but the interpolating motion M(t) is of order $2m$ only.

Figure 5a) shows a rational motion of order 4 interpolating 3 positions with equidistant parameters. Some positions of a parabola in the moving space are drawn additionally. Figure 5b) shows the rational sweeping surface which is generated by moving this parabola $\mathbf{x}(v)$ through the fixed space E^3. This surface is called a *surface of motion* or a *kinematic surface*, cf. [14].

Figure 5: Interpolation with a rational motion of order 4 (a).
A rational surface of motion (b).

It has the parametric representation $\mathbf{y}$(t,v) = M(t)$\mathbf{x}$(v), from which a representation as a rational tensor–product Bézier surface can be computed. The parameter lines t = const of the surface are superposable.

6. Some remarks on spline motions

In order to construct motions interpolating a large number of given positions, rational spline motions (i.e., piecewise rational motions with a certain order of continuity) have to be applied. By replacing the Bernstein polynomials in definition (13) with B–spline basis functions over an appropriate knot sequence (see [7]), the definition of a *B–spline–Q–motion* is obtained. The methods for interpolation with Q–motions presented in this paper can be directly generalized to B–spline–Q–motions. Instead of equation (29), product formulae for B–spline basis functions have to be used, see [10].

Figure 6: A rational C^1–spline–Q–motion

Of course, a lot of constructions for *geometric spline motions* (analogous to geometric spline curves, see [7]) can be derived. As an example, Figure 6 shows a geometric C^1–spline–Q–motion interpolating 6 given positions with equidistant parameters. This motion has been constructed with help of the following

Algorithm 2.

- Given: $m+1$ positions P_i with parameters t_i, see (14) $(i=0,\dots,m)$.

1.) Estimate the velocities $\vec{\mathbf{v}}_i$ of the origin and the angular velocities $\vec{\omega}_i$ of the moving space at the given positions P_i (i=0,...,m)! These velocities and angular velocities can be computed from the translations and rotations connecting the translational and the rotational part of the position P_i with those of the neighbouring positions, respectively.

2.) Construct a rotational cubic Q–motion $Q^{(i)}_{\mathrm{rot}}(t)$ $(i=0,\dots,m-1)$ interpolating the rotational parts of the positions P_i, P_{i+1} and the estimated angular velocities at these positions, see (44)!

3.) Construct a translational cubic Q–motion $Q^{(i)}_{\mathrm{trans}}(t)$ $(i=0,\dots,m-1)$ interpolating the translational parts of the positions P_i, P_{i+1} and the estimated velocities of the origin at these positions, see (41)!

4.) The whole motion is defined piecewise as the composition of the translational and the rotational interpolant:

$$Q(t) = Q^{(i)}_{\mathrm{trans}}(t) * Q^{(i)}_{\mathrm{rot}}(t) \quad \text{for} \quad t \in [t_i, t_{i+1}). \tag{40}$$

The third step of this algorithm is analogous to the construction of a cubic C^1–spline curve. The i–th segment of the translational spline Q–motion is given by

$$\begin{aligned} Q^{(i)}_{\mathrm{trans}}(t) &= 2 + \varepsilon \sum_{j=0}^{3} b^3_j\left(\tfrac{t-t_i}{t_{i+1}-t_i}\right) \vec{\mathbf{p}}^{(i)}_j \\ \text{where} \quad \vec{\mathbf{p}}^{(i)}_0 &= \vec{\mathbf{s}}_i \quad , \quad \vec{\mathbf{p}}^{(i)}_1 = \vec{\mathbf{s}}_i + \tfrac{t_{i+1}-t_i}{3} \vec{\mathbf{v}}_i, \\ \vec{\mathbf{p}}^{(i)}_2 &= \vec{\mathbf{s}}_{i+1} - \tfrac{t_{i+1}-t_i}{3} \vec{\mathbf{v}}_{i+1} \quad \text{and} \quad \vec{\mathbf{p}}^{(i)}_3 = \vec{\mathbf{s}}_{i+1} \end{aligned} \tag{41}$$

$(i=0,\dots,m-1)$. The second step requires the formula

$$\vec{\omega}(t) = 2\,\mathrm{Vec}\,\frac{\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathrm{rot}}(t) * \widetilde{Q}_{\mathrm{rot}}(t)}{Q_{\mathrm{rot}}(t) * \widetilde{Q}_{\mathrm{rot}}(t)} \tag{42}$$

for the angular velocity of a rotational Q–motion (cf. [3, p.521]). From this equation, the condition

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathrm{rot}}(t_i) = \tfrac{1}{2}(R_i * \widetilde{R}_i)\, \vec{\omega}_i * R_i + \lambda R_i \tag{43}$$

for the interpolation of the angular velocity $\vec{\omega}_i$ at a given position with rotational part R_i can be derived directly. The real coefficient λ is arbitrary.

Let again $R^{(0)}_i$ denote the normalized quaternions obtained from the rotational parts of the given positions, cf. (23) and (24). The i–th segment of the rotational spline Q–motion is given by

$$Q_{\rm rot}^{(i)}(t) = 2 + \varepsilon \sum_{j=0}^{3} b_j^3(\tfrac{t-t_i}{t_{i+1}-t_i})\, C_j^{(i)}$$

$$\text{where}\quad C_0^{(i)} = R_i^{(0)}\quad,\qquad C_1^{(i)} = R_i^{(0)} + \tfrac{t_{i+1}-t_i}{6}\,\vec{\omega}_i * R_i^{(0)}, \qquad (44)$$

$$C_2^{(i)} = R_{i+1}^{(0)} - \tfrac{t_{i+1}-t_i}{6}\,\vec{\omega}_{i+1} * R_{i+1}^{(0)} \quad\text{and}\quad C_3^{(i)} = R_{i+1}^{(0)}$$

($i = 0, \ldots, m-1$). Note that the construction of the spline Q–motion works completely *local*, i.e., the construction of the i–th segment requires only the knowledge of a finite number of neighbouring positions which are used in order to estimate the velocities $\vec{\mathbf{v}}_i, \vec{\mathbf{v}}_{i+1}$ and the angular velocities $\vec{\omega}_i, \vec{\omega}_{i+1}$ at the endpoints of the spline segment.
The segments of the above spline motion are Q–motions of polynomial degree 6. Similar to section 4, a C^1–spline motion whose segments are rational motions of order 6 can be constructed, too.

7. The visualization algorithm

The visualization of the moving object is summarized in the following

Algorithm 3.

- Given: Some positions P_i of the moving object with parameters t_i, see (14).

1.) Interpolate the given positions by a Q–motion or a spline–Q–motion obtained from the Algorithms 1 or 2!

2.) Compute the trajectories of the points of the moving object, see (10) and (12)! (Of course, it is sufficient to compute the trajectories of some "key points", those of the remaining points result from linear interpolation.)

3.) `for (t=0; t≤1; t+=△t)`

  - The position of the moving object in 3–space is found by evaluating the trajectories of the key points.
  - Visualize the object using appropriate visibility and shading algorithms!

Conclusion

In this paper, a method for the interpolation of spatial displacements has been derived with help of dual quaternion curves. The trajectories of the moving object under the interpolating

motion are rational Bézier curves, thus the powerful methods of Computer Aided Geometric Design can be applied to these curves. The dependence of the result of the interpolation method on the choice of the coordinate system has been discussed thoroughly.
A detailed introduction to mathematical methods for the computer–aided design of rational motions will be given in [8]. The author thinks that dual quaternion curves have proved to be a very useful tool in Computer Graphics.

References

1. J. Angeles, *Rational Kinematics*, Springer, New York, (1988).
2. W. Blaschke, *Kinematik und Quaternionen*, Deutscher Verlag der Wissenschaften, Berlin (1960).
3. O. Bottema and B. Roth, *Theoretical Kinematics*, North-Holland Publishing Company, Amsterdam, New York, Oxford (1979).
4. G. Darboux, Sur les mouvements algebriques, Note III in G. Kœnigs, G.: *Leçons de cinématique*, Hermann, Paris (1895).
5. Q. J. Ge and J. M. McCarthy, An Algebraic Formulation of Configuration–Space Obstacles for Spatial Robots, *Proc. IEEE International Conference on Robotics and Automation*, 1542–1547 (1990).
6. Q. J. Ge and B. Ravani, Computer Aided Geometric Design of Motion Interpolants, *Proc. ASME Design Automation Conference*, DE-Vol. **32–2**, 33–41 (1991).
7. J. Hoschek and D. Lasser, *Fundamentals of Computer Aided Geometric Design*, Peters, Wellesley (1993).
8. B. Jüttler, *Mathematische Grundlagen für den rechnergestützten Entwurf rationaler Bewegungsvorgänge* Dissertation, TH Darmstadt, to appear (1994).
9. B. Jüttler, Über zwangläufige rationale Bewegungsvorgänge, *Sitzungsber. d. Österr. Akad. d. Wiss.*, accepted for publication (1993).
10. K. Mørken, Some identities for products and degree raising of splines, *Constructive Approximation* **7**, 195–208 (1991).
11. G. Nielson, *Bernstein/Bézier curves and splines on spheres based upon a spherical de Casteljau algorithm*, Arizona State University Technical Report TR–88–028 (1988).
12. D. Pletinckx, Quaternion calculus as a basic tool in computer graphics, *The Visual Computer* **5**, 2–13 (1989).

13. O. Röschel, Rationale räumliche Zwangläufe vierter Ordnung, *Sitzungsber. d. Österr. Akad. d. Wiss.* **194**, 185–202 (1985) (abstracted in Math. Reviews 87k: 53016).

14. O. Röschel, Kinematic rational Bézier patches I, II, *Rad Jugoslav. Akad. Znan. Umjet.*, in print (1993).

15. K. Shoemake, Animating Rotation with Quaternion Curves, *ACM Siggraph* **19**, 245–254 (1985).

16. W. Wunderlich, Kubische Zwangläufe, *Sitzungsber. d. Österr. Akad. d. Wiss.* **193**, 45–68 (1984) (abstracted in Math. Reviews 86h: 53011).